
We recommend you cite the published version. The publisher’s URL is [http://dx.doi.org/10.1016/j.aml.2007.10.016](http://dx.doi.org/10.1016/j.aml.2007.10.016)

Refereed: No

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The \( k \)-Tuple Domination Number Revisited

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Abstract
The following fundamental result for the domination number \( \gamma(G) \) of a graph \( G \) was proved by Alon and Spencer, Arnautov, Lovász and Payan:
\[
\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1}n,
\]
where \( n \) is the order and \( \delta \) is the minimum degree of vertices of \( G \). A similar upper bound for the double domination number was found by Harant and Henning [On double domination in graphs. Discuss. Math. Graph Theory 25 (2005) 29–34], and for the triple domination number by Rautenbach and Volkmann [New bounds on the \( k \)-domination number and the \( k \)-tuple domination number. Applied Math. Letters 20 (2007) 98–102], who also posed the interesting conjecture on the \( k \)-tuple domination number: for any graph \( G \) with \( \delta \geq k - 1 \),
\[
\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\hat{d}_{k-1} + \hat{d}_{k-2}) + 1}{\delta - k + 2}n,
\]
where \( \hat{d}_m = \sum_{i=1}^n \left( \frac{d_i}{m} \right) /n \) is the \( m \)-degree of \( G \). This conjecture, if true, would generalise all the mentioned upper bounds and improve an upper bound proved in [A. Gagarin and V. Zverovich, A generalised upper bound for the \( k \)-tuple domination number. Discrete Math. (to appear)].

In this paper, we prove Rautenbach–Volkmann’s conjecture.

Keywords: graphs, domination number, double domination, triple domination, \( k \)-tuple domination.

1 Notation
All graphs will be finite and undirected without loops and multiple edges. If \( G \) is a graph of order \( n \), then \( V(G) = \{v_1, v_2, ..., v_n\} \) is the set of vertices in \( G \), \( d_i \) denotes the degree of \( v_i \) and \( d = \sum_{i=1}^n d_i/n \) is the average degree of \( G \). Let \( N(x) \) denote the neighbourhood of a vertex \( x \). Also let \( N(X) = \bigcup_{x \in X} N(x) \) and \( N[X] = N(X) \cup X \). Denote by \( \delta(G) \) and \( \Delta(G) \) the minimum and maximum degrees of vertices of \( G \), respectively. Put \( \delta = \delta(G) \) and \( \Delta = \Delta(G) \). A set \( X \) is called a dominating set if every vertex not in \( X \) is adjacent to a vertex in \( X \). The minimum cardinality of a dominating set of \( G \) is the domination number \( \gamma(G) \). A set \( X \) is called a \( k \)-tuple dominating set of \( G \) if for every vertex \( v \in V(G) \), \( |N[v] \cap X| \geq k \). The minimum cardinality of a \( k \)-tuple dominating set of \( G \) is the \( k \)-tuple domination number \( \gamma_{\times k}(G) \). The \( k \)-tuple domination number is only defined for graphs with \( \delta \geq k - 1 \). It is easy to see that \( \gamma(G) = \gamma_{\times 1}(G) \) and \( \gamma_{\times k}(G) \leq \gamma_{\times k'}(G) \) for \( k \leq k' \). The 2-tuple domination number \( \gamma_{\times 2}(G) \) is called the double domination number and the 3-tuple domination number \( \gamma_{\times 3}(G) \) is called the triple domination number. A number of interesting results on the \( k \)-tuple domination number can be found in [3]–[8] and [11].
2 Introduction

The following fundamental result was proved by many authors:

**Theorem 1 ([1, 2, 9, 10])** For any graph $G$, 
\[
\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1}n.
\]

A similar upper bound for the double domination number was found by Harant and Henning [4]:

**Theorem 2 ([4])** For any graph $G$ with $\delta \geq 1$, 
\[
\gamma_{\times 2}(G) \leq \frac{\ln \delta + \ln(d + 1) + 1}{\delta}n.
\]

Rautenbach and Volkmann posed the following interesting conjecture for the $k$-tuple domination number:

**Conjecture 1 ([11])** For any graph $G$ with $\delta \geq k - 1$, 
\[
\gamma_{\times k}(G) \leq \frac{\ln (\delta - k + 2) + \ln \left( \sum_{i=1}^{n} \left( \frac{d_i + 1}{k - 1} \right) \right) - \ln(n) + 1}{\delta - k + 2}n.
\]

For $m \leq \delta$, let us define the $m$-degree $\hat{d}_m$ of a graph $G$ as follows:
\[
\hat{d}_m = \hat{d}_m(G) = \sum_{i=1}^{n} \left( \frac{d_i}{m} \right) / n.
\]

Note that $\hat{d}_1$ is the average degree $d$ of a graph and $\hat{d}_0 = 1$. Also, we put $\hat{d}_{-1} = 0$.

Since 
\[
\left( \frac{d_i + 1}{k - 1} \right) = \left( \frac{d_i}{k - 1} \right) + \left( \frac{d_i}{k - 2} \right),
\]
we see that the above conjecture can be re-formulated as follows:

**Conjecture 1′** For any graph $G$ with $\delta \geq k - 1$, 
\[
\gamma_{\times k}(G) \leq \frac{\ln (\delta - k + 2) + \ln (\hat{d}_{k-1} + \hat{d}_{k-2}) + 1}{\delta - k + 2}n.
\]

It may be pointed out that this conjecture, if true, would generalise Theorem 2 and also Theorem 1 taking into account that $\hat{d}_{-1} = 0$. Rautenbach and Volkmann proved the above conjecture for the triple domination number:

**Theorem 3 ([11])** For any graph $G$ with $\delta \geq 2$, 
\[
\gamma_{\times 3}(G) \leq \frac{\ln(\delta - 1) + \ln(\hat{d}_2 + d) + 1}{\delta - 1}n.
\]

The next result generalises all the above theorems, but it is still far from Conjecture 1′.

**Theorem 4 ([3])** For any graph $G$ with $\delta \geq k - 1$, 
\[
\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln \left( \sum_{m=1}^{k-1} (k - m)\hat{d}_m + \epsilon \right) + 1}{\delta - k + 2}n,
\]
where $\epsilon = 1$ if $k = 1$ or $2$, and $\epsilon = -d$ if $k \geq 3$. 

2
3 Proof of the Conjecture

The following theorem proves Rautenbach–Volkmann’s conjecture.

**Theorem 5** For any graph $G$ with $\delta \geq k - 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\hat{d}_{k-1} + \hat{d}_{k-2}) + 1}{\delta - k + 2} n.$$ 

**Proof:** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability $p$, $0 \leq p \leq 1$. For $m = 0, 1, ..., k - 1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$ 

Also, for $m = 0, 1, ..., k - 2$, we denote

$$A_m = \{v_i \in A : |N(v_i) \cap A| = m\}.$$ 

For each set $A_m$, we form a set $A'_m$ in the following way. For every vertex in the set $A_m$, we take $k - m - 1$ neighbours not in $A$ and add them to $A'_m$. Such neighbours always exist because $\delta \geq k - 1$. It is obvious that $|A'_m| \leq (k - m - 1)|A_m|$. For each set $B_m$, we form a set $B'_m$ by taking $k - m - 1$ neighbours not in $A$ for every vertex in $B_m$. We have $|B'_m| \leq (k - m - 1)|B_m|$.

We construct the set $D$ as follows:

$$D = A \cup \left( \bigcup_{m=0}^{k-2} A'_m \right) \cup \left( \bigcup_{m=0}^{k-1} B_m \cup B'_m \right).$$

The set $D$ is a $k$-tuple dominating set. Indeed, if there is a vertex $v$ which is not $k$-tuple dominated by $D$, then $v$ is not $k$-tuple dominated by $A$. Therefore, $v$ would belong to $A_m$ or $B_m$ for some $m$, but all such vertices are $k$-tuple dominated by the set $D$ by construction.

The expected value of $|D|$ is

$$E(|D|) \leq E\left(|A| + \sum_{m=0}^{k-2} |A'_m| + \sum_{m=0}^{k-1} |B_m| + \sum_{m=0}^{k-1} |B'_m|\right).$$

$$\leq E\left(|A| + \sum_{m=0}^{k-2} (k - m - 1)|A_m| + \sum_{m=0}^{k-1} (k - m)|B_m|\right).$$

$$= E(|A|) + \sum_{m=0}^{k-2} (k - m - 1)E(|A_m|) + \sum_{m=0}^{k-1} (k - m)E(|B_m|).$$

We have

$$E(|A|) = \sum_{i=1}^{n} P(v_i \in A) = pn.$$ 

Also,

$$E(|A_m|) = \sum_{i=1}^{n} P(v_i \in A_m) = \sum_{i=1}^{n} p \binom{d_i}{m} p^m (1 - p)^{d_i - m}$$ \[ \leq p^{m+1}(1 - p)^{\delta - m} \sum_{i=1}^{n} \binom{d_i}{m} \]

$$= p^{m+1}(1 - p)^{\delta - m} \hat{d}_m n.$$
and
\[
E(|B_m|) = \sum_{i=1}^{n} P(v_i \in B_m)
\]
\[
= \sum_{i=1}^{n} (1 - p) \left( \frac{d_i}{m} \right) p^m (1 - p)^{d_i - m}
\]
\[
\leq p^m (1 - p)^{\delta - m + 1} \sum_{i=1}^{n} \left( \frac{d_i}{m} \right)
\]
\[
= p^m (1 - p)^{\delta - m + 1} \hat{d}_m n.
\]

Taking into account that \( \hat{d}_{-1} = 0 \), we obtain
\[
E(|D|) \leq pn + \sum_{m=0}^{k-2} (k - m - 1)p^{m+1}(1 - p)^{\delta - m} \hat{d}_m n + \sum_{m=0}^{k-1} (k - m)p^m (1 - p)^{\delta - m + 1} \hat{d}_m n
\]
\[
=pn + \sum_{m=1}^{k-1} (k - m)p^m (1 - p)^{\delta - m + 1} \hat{d}_{m-1} n + \sum_{m=0}^{k-1} (k - m)p^m (1 - p)^{\delta - m + 1} \hat{d}_m n
\]
\[
= pn + \sum_{m=0}^{k-1} (k - m)p^m (1 - p)^{\delta - m + 1} \hat{d}_{m-1} + \hat{d}_m)n
\]
\[
= pn + (1 - p)^{\delta - k + 2} n \sum_{m=0}^{k-1} (k - m)p^m (1 - p)^{k-m-1} (\hat{d}_{m-1} + \hat{d}_m).
\]

Let us denote
\[
\mu = \delta - k + 2.
\]

Using the inequality \( 1 - x \leq e^{-x} \), we obtain
\[
(1 - p)^{\delta - k + 2} = (1 - p)^{\mu} \leq e^{-p\mu}.
\]

Thus,
\[
E(|D|) \leq pn + e^{-p\mu} n \Theta,
\]

where
\[
\Theta = \sum_{m=0}^{k-1} (k - m)p^m (1 - p)^{k-m-1} (\hat{d}_m + \hat{d}_{m-1}). \tag{1}
\]

We will prove that
\[
\Theta \leq \hat{d}_{k-1} + \hat{d}_{k-2}.
\]

We have
\[
\Theta = \sum_{m=0}^{k-1} (k - m)(\hat{d}_m + \hat{d}_{m-1}) \sum_{i=0}^{k-m-1} (-1)^i \binom{k-m-1}{i} p^{m+i}
\]
\[
= k(\hat{d}_0 + \hat{d}_{-1}) \binom{k-1}{0} p^0 - k(\hat{d}_0 + \hat{d}_{-1}) \binom{k-1}{1} p^1 + \ldots + k(\hat{d}_0 + \hat{d}_{-1}) \binom{k-1}{k-1} (-1)^{k-1} p^{k-1}
\]
\[
+ (k - 1)(\hat{d}_1 + \hat{d}_0) \binom{k-2}{0} p^1 + \ldots + (k - 1)(\hat{d}_1 + \hat{d}_0) \binom{k-2}{k-2} (-1)^{k-2} p^{k-1}
\]
\[
\ldots
\]
\[
\ldots
\]
Let us denote

\[ s_j = \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} (i+j+1)(\hat{d}_{k-i-j-1} + \hat{d}_{k-i-j-2}) \]

(taking into account that \( \hat{d}_{-1} = 0 \))

\[ = \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} (i+j+1)\hat{d}_{k-i-j-1} + \sum_{i=0}^{k-j-2} (-1)^i \binom{i+j}{i} (i+j+1)\hat{d}_{k-i-j-2} \]

\[ = \binom{j}{0} (j+1)\hat{d}_{k-j-1} + \sum_{i=1}^{k-j-1} (-1)^i \binom{i+j}{i} (i+j+1)\hat{d}_{k-i-j-1} \]

\[ + \sum_{i=1}^{k-j-1} (-1)^{i-1} \binom{i+j-1}{i-1} (i+j)\hat{d}_{k-i-j-1} \]

\[ = (j+1)\hat{d}_{k-j-1} + \sum_{i=1}^{k-j-1} (-1)^i (j+1) \binom{i+j}{i} \hat{d}_{k-i-j-1} \]

\[ = (j+1) \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} \hat{d}_{k-i-j-1} \]

\[ = (j+1) \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} \sum_{l=1}^{n} \binom{d_l}{k-i-j-1} / n \]

\[ = (j+1) \sum_{l=1}^{n} \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} \frac{d_l}{k-i-j-1} / n \]

\[ = (j+1) \sum_{l=1}^{n} \frac{d_l - j - 1}{k - j - 1} / n \] (by Lemma 3)

\[ \geq 0. \]

Thus, the function \( \Theta(p) = s_0 p^{k-1} + s_1 p^{k-2} + \ldots + s_{k-1} \) is monotonically increasing in \( 0 \leq p \leq 1 \).

Therefore, (1) implies

\[ \Theta \leq \hat{d}_{k-1} + \hat{d}_{k-2}. \]

We obtain

\[ E(|D|) \leq pn + e^{-pn}n\Theta \leq pn + e^{-pn}(\hat{d}_{k-1} + \hat{d}_{k-2}). \]

Let us denote

\[ f(p) = pn + e^{-pn}(\hat{d}_{k-1} + \hat{d}_{k-2}). \]

For \( p \in [0, 1] \), the function \( f(p) \) is minimised at the point \( \min\{1, z\} \), where

\[ z = \frac{\ln \mu + \ln(\hat{d}_{k-1} + \hat{d}_{k-2})}{\mu}. \]
There are two cases to consider.

If \( z \leq 1 \), then

\[
E(|D|) \leq f(z) = \left( z + \frac{1}{\mu} \right) n = \frac{\ln \mu + \ln(\hat{d}_{k-1} + \hat{d}_{k-2}) + 1}{\mu} n.
\]

Since the expected value is an average value, there exists a particular \( k \)-tuple dominating set of order at most \( f(z) \), as required.

Suppose now that \( z > 1 \). Taking into account that \( \mu > 0 \), we obtain

\[
\gamma_{\times k}(G) \leq n < \left( z + \frac{1}{\mu} \right) n = \frac{\ln \mu + \ln(\hat{d}_{k-1} + \hat{d}_{k-2}) + 1}{\mu} n,
\]

as required. The proof of Theorem 5 is complete.

For \( s \geq 1 \), let us denote

\[
T_s^t = \binom{s}{t} - \binom{s}{t-1} + \ldots + (-1)^i \binom{s}{0}.
\]

**Lemma 1**

\[
T_s^t = \binom{s-1}{t}.
\]

**Proof:** Induction on \( t \):

\[
T_s^t = \binom{s}{t} - T_s^{t-1} = \binom{s}{t} - \binom{s-1}{t-1} = \binom{s-1}{t}.
\]

**Lemma 2** For \( j \geq 1 \),

\[
\binom{j-1}{0} + \binom{j}{1} + \ldots + \binom{j+i-1}{i} = \binom{j+i}{i}.
\]

**Proof:** Induction on \( i \):

\[
\binom{j-1}{0} + \binom{j}{1} + \ldots + \binom{j+i-1}{i} = \binom{j+i-1}{i-1} + \binom{j+i-1}{i} = \binom{j+i}{i}.
\]

**Lemma 3**

\[
\sum_{i=0}^{l} (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \binom{r-j-1}{l}.
\]

**Proof:** Induction on \( j \). If \( j = 0 \), then

\[
\sum_{i=0}^{l} (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \sum_{i=0}^{l} (-1)^i \binom{r}{l-i} = T_l^r = \binom{r-1}{l},
\]

as required.

Suppose that \( j \geq 1 \) and the equation of Lemma 3 is true for any \( j' \leq j-1 \). Applying Lemmas 1 and 2, we obtain:
\[
\sum_{i=0}^{l} (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \sum_{i=0}^{l} (-1)^i \left( \binom{j-1}{0} + \binom{j}{1} + \cdots + \binom{j+i-1}{j} \right) \binom{r}{l-i} \\
= \left( \binom{j-1}{0} \right) \sum_{i=0}^{l} (-1)^i \binom{r}{l-i} + \left( \binom{j}{1} \sum_{i=1}^{l} (-1)^i \binom{r}{l-i} \right) + \cdots \] \\
= \left( \binom{j-1}{0} \right) T_{l}^{r} - \left( \binom{j}{1} \right) T_{l-1}^{r} + \cdots + \left( \binom{j+l-1}{l} \right) (-1)^l T_{0}^{r} \\
= \sum_{i=0}^{l} (-1)^i \binom{j+i-1}{i} T_{l-i}^{r} \\
= \sum_{i=0}^{l} (-1)^i \binom{j+i-1}{i} \binom{r-1}{l-i} \\
= \binom{r-j-1}{l} \text{.} \quad \text{(by hypothesis)}
\]

References