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Upper bounds for the bondage number of graphs on topological surfaces

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Abstract

The bondage number $b(G)$ of a graph $G$ is the smallest number of edges of $G$ whose removal results in a graph having the domination number larger than that of $G$. We show that, for a graph $G$ having the maximum vertex degree $\Delta(G)$ and embeddable on an orientable surface of genus $h$ and a non-orientable surface of genus $k$,

$$b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}.$$ 

This generalizes known upper bounds for planar and toroidal graphs, and can be improved for bigger values of the genera $h$ and $k$ by adjusting the proofs.

Key words: Bondage number, Domination number, Topological surface, Embedding on a surface, Euler’s formula

1. Introduction

We consider simple finite non-empty graphs. For a graph $G$, its vertex and edge sets are denoted, respectively, by $V(G)$ and $E(G)$. We also use the following standard notation: $d(v)$ for the degree of a vertex $v$ in $G$, $\Delta = \Delta(G)$ for the maximum vertex degree of $G$, $\delta = \delta(G)$ for the minimum vertex degree of $G$, and $N(v)$ for the neighbourhood of a vertex $v$ in $G$. 

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A set $D \subseteq V(G)$ is a dominating set if every vertex not in $D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma(G)$. Clearly, for any spanning subgraph $H$ of $G$, $\gamma(H) \geq \gamma(G)$. The bondage number of $G$, denoted by $b(G)$, is the minimum cardinality of a set of edges $B \subseteq E(G)$ such that $\gamma(G - B) > \gamma(G)$, where $V(G - B) = V(G)$ and $E(G - B) = E(G) \setminus B$. In a sense, the bondage number $b(G)$ measures integrity and reliability of the domination number $\gamma(G)$ with respect to the edge removal from $G$, which may correspond, e.g., to link failures in communication networks.

The bondage number was introduced by Bauer et al. [1] (see also Fink et al. [4]). Two unsolved classical conjectures for the bondage number of arbitrary and planar graphs are as follows.

**Conjecture 1** (Teschner [9]). For any graph $G$, $b(G) \leq \frac{3}{2}\Delta(G)$.

Hartnell and Rall [6] and Teschner [10] showed that for the cartesian product $G_n = K_n \times K_n$, $n \geq 2$, the bound of Conjecture 1 is sharp, i.e. $b(G_n) = \frac{3}{2}\Delta(G_n)$. Teschner [9] also proved that Conjecture 1 holds when $\gamma(G) \leq 3$.

**Conjecture 2** (Dunbar et al. [3]). If $G$ is a planar graph, then $b(G) \leq \Delta(G) + 1$.

The planar graphs are precisely the graphs that can be drawn on the sphere with no crossing edges. A topological surface $S$ can be obtained from the sphere $S_0$ by adding a number of handles or crosscaps. If we add $h$ handles to $S_0$, we obtain an orientable surface $S_h$, which is often referred to as the $h$-holed torus. The number $h$ is called the orientable genus of $S_h$. If we add $k$ crosscaps to the sphere $S_0$, we obtain a non-orientable surface $N_k$. The number $k$ is called the non-orientable genus of $N_k$. Any topological surface is homeomorphically equivalent either to $S_h$ ($h \geq 0$), or to $N_k$ ($k \geq 1$). For example, $S_1, N_1, N_2$ are the torus, the projective plane, and the Klein bottle, respectively.

A graph $G$ is embeddable on a topological surface $S$ if it admits a drawing on the surface with no crossing edges. Such a drawing of $G$ on the surface $S$ is called an embedding of $G$ on $S$. Notice that there can be many different embeddings of the same graph $G$ on a particular surface $S$. The embeddings can be distinguished and classified by different properties. The set of faces of a particular embedding of $G$ on $S$ is denoted by $F(G)$. 

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An embedding of $G$ on the surface $S$ is a 2-cell embedding if each face of the embedding is homeomorphic to an open disk. In other words, a 2-cell embedding is an embedding on $S$ that “fits” the surface. This is expressed in Euler’s formulae (1) and (2) of Theorem 3. For example, a cycle $C_n$ ($n \geq 3$) does not have a 2-cell embedding on the torus, but it has 2-cell embeddings on the sphere and the projective plane. Similarly, a planar graph may have 2-cell and non-2-cell embeddings on the torus.

The following result is usually known as (generalized) Euler’s formula. We state it here in a form similar to Thomassen [11].

**Theorem 3** (Euler’s Formula, [11]). Suppose a connected graph $G$ with $|V(G)|$ vertices and $|E(G)|$ edges admits a 2-cell embedding having $|F(G)|$ faces on a topological surface $S$. Then, either $S = S_h$ and

$$|V(G)| - |E(G)| + |F(G)| = 2 - 2h,$$

(1)

or $S = N_k$ and

$$|V(G)| - |E(G)| + |F(G)| = 2 - k.$$  

(2)

Equation (1) is usually referred to as Euler’s formula for an orientable surface $S_h$ of genus $h$, $h \geq 0$, and Equation (2) is known as Euler’s formula for a non-orientable surface $N_k$ of genus $k$, $k \geq 1$.

The orientable genus of a graph $G$ is the smallest integer $h = h(G)$ such that $G$ admits an embedding on an orientable topological surface $S$ of genus $h$. The non-orientable genus of $G$ is the smallest integer $k = k(G)$ such that $G$ can be embedded on a non-orientable topological surface $S$ of genus $k$. Clearly, in general, $h(G) \neq k(G)$, and the embeddings on $S_{h(G)}$ and $N_{k(G)}$ must be 2-cell embeddings.

Trying to prove Conjecture 2, Kang and Yuan [7] came up with the following upper bound whose simpler topological proof was later discovered by Carlson and Develin [2].

**Theorem 4** ([7, 2]). For any connected planar graph $G$,

$$b(G) \leq \min\{8, \Delta(G) + 2\}.$$

This solves Conjecture 2 in case $\Delta(G) \geq 7$. The upper bound of Theorem 4 is for the sphere $S_0$ that has orientable genus $h = 0$. The proof of Theorem 4 in [2] is topologically intuitive, uses Euler’s formula for the sphere, and allows its authors to establish a partially similar result for the torus.
Theorem 5 ([2]). For any connected toroidal graph $G$, $b(G) \leq \Delta(G) + 3$.

Notice that the torus $S_1$ has orientable genus $h = 1$. As mentioned in [2], it is sufficient to prove the results of Theorems 4 and 5 for connected graphs because the bondage number of a disconnected graph $G$ is the minimum of the bondage numbers of its components.

In this paper, we prove the following result which generalizes the corresponding upper bounds of Theorems 4 and 5 for any orientable or non-orientable topological surface $S$.

Theorem 6. For a connected graph $G$ of orientable genus $h$ and non-orientable genus $k$,

$$b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}.$$ 

The upper bound of Theorem 6 follows from Theorems 8 and 9 proved below in Section 2, and can be improved for bigger values of the genera $h$ and $k$ by adjusting the proofs.

2. The bondage number on orientable and non-orientable surfaces

In this section, we prove Theorem 6 by considering orientable and non-orientable surfaces separately. The proofs are done by using Euler’s formulae (1) and (2), counting arguments, and the following result.

Lemma 7 (Hartnell and Rall [6]). For any edge $uv$ in a graph $G$, we have $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|$. In particular, this implies that $b(G) \leq \delta(G) + \Delta(G) - 1$ (see also [1, 4]).

Having a graph $G$ embedded on a surface $S$, each edge $e_i = uv \in E(G)$, $i = 1, \ldots, |E(G)|$, can be assigned two weights, $w_i = \frac{1}{d(u)} + \frac{1}{d(v)}$ and $f_i = \frac{1}{m'} + \frac{1}{m''}$, where $m'$ is the number of edges on the boundary of a face on one side of $e_i$, and $m''$ is the number of edges on the boundary of the face on the other side of $e_i$. Notice that, in an embedding on a surface, an edge $e_i$ may be not separating two distinct faces, but instead it can appear twice on the boundary of the same face. For example, every edge of a path $P_n$ ($n \geq 2$) embedded on the sphere is on the boundary of a unique face, and it appears exactly twice on the face boundary walk: once for each side of the edge. Clearly, in this case, $m' = m'' = 2(n - 1)$ and $f_i = \frac{2}{m'} = \frac{2}{m''} = \frac{1}{n-1}$.
Notice that weights $w_i$ and $f_i$, $i = 1, \ldots, |E(G)|$, count the number of vertices of $G$ and faces of its embedding on $S$ as follows:

\[
\sum_{i=1}^{|E(G)|} w_i = |V(G)|, \quad \sum_{i=1}^{|E(G)|} f_i = |F(G)|.
\]

Then, by Euler’s formula (1), we have

\[
\sum_{i=1}^{|E(G)|} (w_i + f_i - 1) = |V(G)| + |F(G)| - |E(G)| = 2 - 2h,
\]
or, in other words,

\[
\sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 - \frac{2 - 2h}{|E(G)|} \right) = \sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 + \frac{2h - 2}{|E(G)|} \right) = 0.
\]

Now, each edge $e_i = uv \in E(G)$, $i = 1, \ldots, |E(G)|$, can be associated with the quantity $w_i + f_i - 1 + \frac{2h - 2}{|E(G)|}$ called the oriented curvature of the edge. Also, by Euler’s formula (2), we have

\[
\sum_{i=1}^{|E(G)|} (w_i + f_i - 1) = |V(G)| + |F(G)| - |E(G)| = 2 - k,
\]
or, in other words,

\[
\sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 - \frac{2 - k}{|E(G)|} \right) = \sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 + \frac{k - 2}{|E(G)|} \right) = 0.
\]

Then, each edge $e_i = uv \in E(G)$, $i = 1, \ldots, |E(G)|$, can be associated with the quantity $w_i + f_i - 1 + \frac{k - 2}{|E(G)|}$ called the non-oriented curvature of the edge.

**Theorem 8.** Let $G$ be a connected graph 2-cell embeddable on an orientable surface of genus $h \geq 0$. Then

\[
b(G) \leq \Delta(G) + h + 2.
\]
PROOF. Suppose $G$ is 2-cell embedded on the $h$-holed torus $S_h$. By Lemma 7, if $G$ has any vertices of degree $h+3$ or less, we have $\delta(G) \leq h+3$, and inequality (3) holds. Therefore, we can assume $\Delta(G) \geq \delta(G) \geq h+4$.

Now, suppose the opposite, $b(G) \geq \Delta(G) + h + 3$. Then, by Lemma 7, for any edge $e_i = uv, i = 1, \ldots, |E(G)|$, we have

$$d(u) + d(v) - 1 - |N(u) \cap N(v)| \geq b(G) \geq \Delta(G) + h + 3.$$ 

This gives

$$d(u) + d(v) \geq \Delta(G) + h + 4 + |N(u) \cap N(v)|,$$

and $d(u) \leq \Delta(G), d(v) \leq \Delta(G)$. If either $d(u)$ or $d(v)$ is equal to $h+4$, then, by (4), the other degree must be equal to $\Delta(G) \geq h+4$, and $u$ and $v$ cannot have any common neighbors, so that $m'$ and $m''$ are at least 4 each. Since in this case $|E(G)| \geq \frac{(h+4)(h+5)}{2}$, such an edge $e_i = uv$ has a negative oriented curvature:

$$w_i + f_i - 1 + \frac{2h-2}{|E(G)|} \leq \frac{2}{h+4} + \frac{2}{4} - 1 + \frac{2(2h-2)}{(h+4)(h+5)} = \frac{-8 + h(3-h)}{2(h+4)(h+5)} < 0$$

for any $h \geq 1$, and, in case $h = 0$,

$$w_i + f_i - 1 - \frac{2}{|E(G)|} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 - \frac{2}{|E(G)|} = \frac{-2}{|E(G)|} < 0.$$ 

Suppose one of $d(u)$ and $d(v)$ is equal to $h+5$, without loss of generality, $d(u) = h+5$. Then, by (4), $\Delta(G) \geq d(v) \geq \Delta(G) - 1 + |N(u) \cap N(v)|$.

If $d(v) = h+4 = \Delta(G) - 1$, we are in the previous case. Otherwise, we have $d(v) \geq h+5$, and, by (4), at most one of $m'$ and $m''$ can be equal to 3, implying the other is at least 4. Then again, since in this case $|E(G)| \geq \frac{(h+4)(h+4)+2(h+5)}{2} = \frac{h^2 + 10h + 26}{2}$, the edge $e_i$ must have a negative oriented curvature:

$$w_i + f_i - 1 + \frac{2h-2}{|E(G)|} \leq \frac{2}{h+5} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} - 1 + \frac{2(2h-2)}{h^2 + 10h + 26} = \frac{-5h^3 - 3h^2 + 52h - 266}{12(h+5)(h^2 + 10h + 26)} < 0$$

for any $h \geq 1$, and, in case $h = 0$,

$$w_i + f_i - 1 - \frac{2}{|E(G)|} \leq \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{4} - 1 - \frac{2}{|E(G)|} = \frac{1}{60} - \frac{2}{|E(G)|} < 0.$$
The only remaining case is when \( d(u) \geq h + 6 \) and \( d(v) \geq h + 6 \). Since \( m' \geq 3 \) and \( m'' \geq 3 \), and, in this case, \(|E(G)| \geq \frac{(h+4)(h+5)+2(h+6)}{2} = \frac{h^2+11h+32}{2}\), the edge \( e_i \) must have a negative oriented curvature:

\[
\frac{w_i + f_i - 1 + \frac{2h - 2}{|E(G)|}}{2} \leq \frac{2}{h + 6} + 1 + \frac{2(2h - 2)}{h^2 + 11h + 32} = \frac{-h^3 + h^2 + 28h - 72}{3(h + 6)(h^2 + 11h + 32)} < 0
\]

for any \( h \geq 1 \), and, in case \( h = 0 \),

\[
\frac{w_i + f_i - 1 - \frac{2}{|E(G)|}}{2} \leq \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} - 1 - \frac{2}{|E(G)|} = \frac{-2}{|E(G)|} < 0.
\]

Summing over all edges \( e_i \in E(G) \) yields

\[
\sum_{i=1}^{|E(G)|} \left( \frac{w_i + f_i - 1 + \frac{2h - 2}{|E(G)|}}{2} \right) < 0,
\]

which is a contradiction to Euler’s formula (1) stating

\[
\sum_{i=1}^{|E(G)|} \left( \frac{w_i + f_i - 1}{|E(G)|} \right) = |V(G)| + |F(G)| - |E(G)| - (2 - 2h) = 0.
\]

Thus, \( b(G) \leq \Delta(G) + h + 2 \).

\(\square\)

**Theorem 9.** Let \( G \) be a connected graph 2-cell embeddable on a non-orientable surface of genus \( k \geq 1 \). Then

\[
b(G) \leq \Delta(G) + k + 1. \tag{5}
\]

**Proof.** Suppose \( G \) is 2-cell embedded on the sphere with \( k \) crosscaps \( N_k \). By Lemma 7, if \( G \) has any vertices of degree \( k + 2 \) or less, we have \( \delta(G) \leq k + 2 \), and inequality (5) holds. Therefore, we can assume \( \Delta(G) \geq \delta(G) \geq k + 3 \).

Suppose the opposite, \( b(G) \geq \Delta(G) + k + 2 \). Then, by Lemma 7, for any edge \( e_i = uv, i = 1, \ldots, |E(G)| \), we have \( d(u) + d(v) - 1 - |N(u) \cap N(v)| \geq b(G) \geq \Delta(G) + k + 2 \). Then, \( d(u) + d(v) \geq \Delta(G) + k + 3 + |N(u) \cap N(v)| \), and \( d(u) \leq \Delta(G), d(v) \leq \Delta(G) \). If either \( d(u) \) or \( d(v) \) is equal to \( k + 3 \), the other degree must be equal to \( \Delta(G) \geq k + 3 \), and \( u \) and \( v \) cannot have any common neighbors, so that \( m' \) and \( m'' \) are at least 4 each. Since in this case \(|E(G)| \geq \frac{(k+3)(k+4)}{2}\), the non-oriented curvature of the edge \( e_i = uv \) is

\[
\frac{w_i + f_i - 1 + \frac{k - 2}{|E(G)|}}{k + 3} \leq \frac{2}{k + 3} + \frac{2}{4} - 1 + \frac{2(k - 2)}{(k + 3)(k + 4)} = \frac{-4 + k(1 - k)}{2(k + 3)(k + 4)} < 0
\]
for any $k \geq 2$, and, in case $k = 1$,
\[
    w_i + f_i - 1 - \frac{1}{|E(G)|} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 - \frac{1}{|E(G)|} = -1 < 0.
\]

Suppose one of $d(u)$ and $d(v)$, let us say $d(u)$, is equal to $k + 4$. Then, $\Delta(G) \geq d(v) \geq \Delta(G) - 1 + |N(u) \cap N(v)|$. If $d(v) = k + 3 = \Delta(G) - 1$, we are in the previous case. Otherwise, we have $d(v) \geq k + 4$, and at most one of $m'$ and $m''$ can be equal to 3, implying the other is at least 4. Then again, since in this case $|E(G)| \geq \frac{(k+3)(k+3)+2(k+4)}{2} = k^2 + \frac{8k + 17}{2}$, the edge $e_i$ must have a negative non-oriented curvature:
\[
    w_i + f_i - 1 + \frac{k - 2}{|E(G)|} \leq \frac{2}{k + 4} \frac{2}{3} + \frac{1}{4} - 1 + \frac{2(k - 2)}{k^2 + 8k + 17} = \frac{-124 - 5k - 12k^2 - 5k^3}{12(k + 4)(k^2 + 8k + 17)} < 0
\]
for any $k \geq 2$, and, in case $k = 1$,
\[
    w_i + f_i - 1 - \frac{1}{|E(G)|} \leq \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{4} - 1 - \frac{1}{|E(G)|} = -\frac{1}{60} - \frac{1}{|E(G)|} < 0.
\]

The only remaining case is when $d(u) \geq k + 5$ and $d(v) \geq k + 5$. Since $m' \geq 3$ and $m'' \geq 3$, and, in this case, $|E(G)| \geq \frac{(k+3)(k+3)+2(k+4)}{2} = k^2 + \frac{9k + 22}{2}$, the edge $e_i$ must have a negative non-oriented curvature:
\[
    w_i + f_i - 1 + \frac{k - 2}{|E(G)|} \leq \frac{2}{k + 5} \frac{2}{3} - 1 + \frac{2(k - 2)}{k^2 + 9k + 22} = \frac{-k^3 - 2k^2 + 5k - 38}{3(k + 5)(k^2 + 9k + 22)} < 0
\]
for any $k \geq 2$, and, in case $k = 1$,
\[
    w_i + f_i - 1 - \frac{1}{|E(G)|} \leq \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} - 1 - \frac{1}{|E(G)|} = -\frac{1}{6} < 0.
\]

Summing over all edges $e_i \in E(G)$ yields
\[
    \sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 + \frac{k - 2}{|E(G)|} \right) < 0,
\]
which is a contradiction to Euler’s formula (2) stating
\[
    \sum_{i=1}^{|E(G)|} \left( w_i + f_i - 1 - \frac{2 - k}{|E(G)|} \right) = |V(G)| + |F(G)| - |E(G)| - (2 - k) = 0.
\]
Thus, $b(G) \leq \Delta(G) + k + 1$, and the proof is complete. \qed
3. Conclusions and final remarks

The upper bound of Theorem 6 provides a hierarchy of upper bounds that eventually may help solving Conjecture 1. However, it can be seen that the bounds of Theorems 8 and 9 are not tight for larger values of the genera $h = h(G)$ and $k = k(G)$. For example, by adjusting respectively the proofs of Theorems 8 and 9, upper bound (3) can be improved to $b(G) \leq \Delta(G) + h + 1$ for $h \geq 8$, to $b(G) \leq \Delta(G) + h$ for $h \geq 11$, etc., and upper bound (5) can be improved to $b(G) \leq \Delta(G) + k$ for $k \geq 3$, to $b(G) \leq \Delta(G) + k - 1$ for $k \geq 6$, etc. It is left to the reader to adjust the proofs and bounds for a particular topological surface of higher genus. The bounds of Theorems 8 and 9 are stated in this form for clarity and simplicity of presentation and proofs for smaller values of $h$ and $k$.

In general, one may try to find certain (linear or sublinear) functions of $h$ and $k$ to improve the bounds of Theorems 8 and 9 by replacing the terms $h + 2$ and $k + 1$, respectively, or to provide asymptotically better bounds. For example, simple asymptotic improvements follow from the upper bounds on the minimum vertex degree of graphs embeddable on topological surfaces: it is known that $\delta(G) \leq \lceil \frac{5 + \sqrt{1 + 48h^2}}{2} \rceil$ for $h \geq 1$, $\delta(G) \leq \lceil \frac{5 + \sqrt{1 + 24k^2}}{2} \rceil$ for $k \geq 2$ (e.g., see Sachs [8]), and $\delta(G) \leq 5$ for a planar or projective-planar graph, i.e. when $h = 0$ or $k = 1$. Then, from Lemma 7, we have $b(G) \leq \Delta(G) + \lceil \frac{3 + \sqrt{1 + 48h^2}}{2} \rceil$ for $h \geq 1$ and $b(G) \leq \Delta(G) + \lceil \frac{3 + \sqrt{1 + 24k^2}}{2} \rceil$ for $k \geq 1$, which are better than bounds (3) for $h \geq 12$ and (5) for $k \geq 8$, respectively. However, for example, an adjusted proof of Theorem 9 gives $b(G) \leq \Delta + k - 411 = \Delta + 53$ for $k = 464$, which is better than $b(G) \leq \Delta(G) + \lceil \frac{3 + \sqrt{1 + 24k^2}}{2} \rceil = \Delta + 54$ in this case. Therefore, adjustments of the proofs of Theorems 8 and 9 can provide better results than some asymptotic improvements by using closed formulae, and it would be interesting to have closed formula or asymptotic improvements providing a certain justification of their quality.

In view of Theorem 4, its proof in [2], and results presented in this paper, it should be reasonable to conjecture that, when $\Delta(G)$ is sufficiently large, the bondage number $b(G)$ is bounded by a certain constant depending only on the properties of topological surfaces where $G$ embeds.

**Conjecture 10.** For a connected graph $G$ of orientable genus $h$ and non-orientable genus $k$, $b(G) \leq \min\{c_h, c'_k, \Delta(G) + o(h), \Delta(G) + o(k)\}$, where $c_h$ and $c'_k$ are constants depending, respectively, on the orientable and non-orientable genera of $G$. 

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Since \( \delta(G) \leq 5 \) for a planar graph \( G \), Fischermann et al. [5] ask whether there exist planar graphs of bondage numbers 6, 7, or 8. A class of planar graphs with the bondage number equal to 6 is shown in [2]. Therefore, in the case of planar graphs, we have \( 6 \leq c_0 \leq 8 \). It would be interesting to have an estimation for the constants \( c_h \) and \( c'_k \) for the torus \( S_1 \), projective plane \( N_1 \), and Klein bottle \( N_2 \).

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