The probabilistic approach to limited packings in graphs

Andrei Gagarin\textsuperscript{a,∗}, Vadim Zverovich\textsuperscript{b}

\textsuperscript{a}Department of Computer Science, Royal Holloway, University of London, Egham, Surrey, TW20 0EX, UK
\textsuperscript{b}University of the West of England, Bristol, BS16 1QY, UK

Abstract

We consider (closed neighbourhood) packings and their generalization in graphs. A vertex set $X$ in a graph $G$ is a $k$-limited packing if for every vertex $v \in V(G)$, $|N[v] \cap X| \leq k$, where $N[v]$ is the closed neighbourhood of $v$. The $k$-limited packing number $L_k(G)$ of a graph $G$ is the largest size of a $k$-limited packing in $G$. Limited packing problems can be considered as secure facility location problems in networks.

In this paper, we develop a new application of the probabilistic method to limited packings in graphs, resulting in lower bounds for the $k$-limited packing number and a randomized algorithm to find $k$-limited packings satisfying the bounds. In particular, we prove that for any graph $G$ of order $n$ with maximum vertex degree $\Delta$,

$$L_k(G) \geq \frac{kn}{(k+1)^k \sqrt{k\binom{\Delta}{k}(\Delta+1)}}.$$

Also, some other upper and lower bounds for $L_k(G)$ are given.

Keywords: $k$-Limited packings, The probabilistic method, Lower and upper bounds, Randomized algorithm

1. Introduction

We consider simple undirected graphs. If not specified otherwise, standard graph-theoretic terminology and notations are used (e.g., see [1, 2]). We are interested in the classical packings and packing numbers of graphs as introduced in [9], and their generalization, called limited packings and limited packing numbers, respectively, as presented in [6]. In the literature, the classical packings are often referred to under different names: for example, as (distance) 2-packings [9, 13], closed neighborhood packings [11] or strong stable sets [8]. They can also be considered as generalizations of independent (stable) sets which, following the terminology of [9], would be (distance) 1-packings.
Formally, a vertex set $X$ in a graph $G$ is a $k$-limited packing if for every vertex $v \in V(G)$, 
\[ |N[v] \cap X| \leq k, \]
where $N[v]$ is the closed neighbourhood of $v$. The $k$-limited packing number $L_k(G)$ of a graph $G$ is the maximum size of a $k$-limited packing in $G$. In these terms, the classical (distance) 2-packings are 1-limited packings, and hence $\rho(G) = L_1(G)$, where $\rho(G)$ is the 2-packing number.

The problem of finding a 2-packing (1-limited packing) of maximum size is shown to be NP-complete by Hochbaum and Schmoys [8]. In [4], it is shown that the problem of finding a maximum size $k$-limited packing is NP-complete even for the classes of split and bipartite graphs.

Graphs usually serve as underlying models for networks. A number of interesting application scenarios of limited packings are described in [6], including network security, market saturation, and codes. These and others can be summarized as secure location or distribution of facilities in a network. In a more general sense, these problems can be viewed as (maximization) facility location problems to place/distribute in a given network as many resources as possible subject to some (security) constraints.

2-Packings (1-limited packings) are well-studied in the literature from the structural and algorithmic point of view (e.g., see [8, 9, 11, 12]) and in connection with other graph parameters (e.g., see [3, 7, 9, 11, 13]). In particular, several papers discuss connections between packings and dominating sets in graphs (e.g., see [3, 4, 6, 7, 11]). Although the formal definitions for packings and dominating sets may appear to be similar, the problems have a very different nature: one of the problems is a maximization problem not to break some (security) constraints, and the other is a minimization problem to satisfy some reliability requirements. For example, given a graph $G$, the definitions imply a simple inequality $\rho(G) \leq \gamma(G)$, where $\gamma(G)$ is the domination number of $G$ (e.g., see [11]). However, the difference between $\rho(G)$ and $\gamma(G)$ can be arbitrarily large as illustrated in [3]: $\rho(K_n \times K_n) = 1$ for the Cartesian product of complete graphs, but $\gamma(K_n \times K_n) = n$.

In this paper, we develop an application of the probabilistic method to $k$-limited packings in general and to 2-packings (1-limited packings) in particular. In Section 2 we present the probabilistic construction and use it to derive two lower bounds for the $k$-limited packing number $L_k(G)$. Also, using a greedy algorithm approach, we provide an improved lower bound for the 2-packing (1-limited packing) number $\rho(G) = L_1(G)$. The probabilistic construction implies a randomized algorithm to find $k$-limited packings satisfying the lower bounds. The algorithm and its analysis are presented in Section 3. Section 4 shows that the main lower bound is asymptotically sharp, and discusses the improvement for 1-limited packings from the greedy algorithm approach. Finally, Section 5 provides upper bounds for $L_k(G)$, e.g. in terms of the $k$-tuple domination number $\gamma_{\times k}(G)$.

Notice that the probabilistic construction and approach are different from the well-known probabilistic constructions used for independent sets (e.g., see [1], p.27–28). In terms of packings, an independent set in a graph $G$ is a distance 1-packing: for any two vertices in an independent set, the distance between them in $G$ is greater
than 1. To the best of our knowledge, the proposed application of the probabilistic method is a new approach to work with packings and related maximization problems.

2. The probabilistic construction and lower bounds

Let $\Delta = \Delta(G)$ denote the maximum vertex degree in a graph $G$. Notice that $L_k(G) = n$ when $k \geq \Delta + 1$. We define

$$c_t = c_t(G) = \binom{\Delta}{t} \quad \text{and} \quad \tilde{c}_t = \tilde{c}_t(G) = \binom{\Delta + 1}{t}.$$ 

In what follows, we put $(a, b) = 0$ if $b > a$.

The following theorem gives a new lower bound for the $k$-limited packing number. It may be pointed out that the probabilistic construction used in the proof of Theorem 1 implies a randomized algorithm for finding a $k$-limited packing set, whose size satisfies the bound of Theorem 1 with a positive probability (see Algorithm 1 in Section 3).

**Theorem 1.** For any graph $G$ of order $n$ with $\Delta \geq k \geq 1$,

$$L_k(G) \geq \frac{kn}{\tilde{c}^{1/k}_{k+1} (1 + k)^{1+1/k}}. \quad (1)$$

**Proof.** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability

$$p = \left( \frac{1}{\tilde{c}_{k+1} (1 + k)} \right)^{1/k}. \quad (2)$$

For $m = k, ..., \Delta$, we denote

$$A_m = \{ v \in A : |N(v) \cap A| = m \}.$$ 

For each set $A_m$, we form a set $A'_m$ in the following way. For every vertex $v \in A_m$, we take $m - (k - 1)$ neighbours from $N(v) \cap A$ and add them to $A'_m$. Such neighbours always exist because $m \geq k$. It is obvious that

$$|A'_m| \leq (m - k + 1)|A_m|.$$ 

For $m = k + 1, ..., \Delta$, let us denote

$$B_m = \{ v \in V(G) - A : |N(v) \cap A| = m \}.$$ 

For each set $B_m$, we form a set $B'_m$ by taking $m - k$ neighbours from $N(v) \cap A$ for every vertex $v \in B_m$. We have

$$|B'_m| \leq (m - k)|B_m|.$$
Let us construct the set $X$ as follows:

$$X = A - \left( \bigcup_{m=k}^{\Delta} A'_m \right) - \left( \bigcup_{m=k+1}^{\Delta} B'_m \right).$$

It is easy to see that $X$ is a $k$-limited packing in $G$. The expectation of $|X|$ is

$$E[|X|] \geq E\left[|A| - \sum_{m=k}^{\Delta} |A'_m| - \sum_{m=k+1}^{\Delta} |B'_m|\right]$$

$$\geq E\left[|A| - \sum_{m=k}^{\Delta} (m - k + 1)|A_m| - \sum_{m=k+1}^{\Delta} (m - k)|B_m|\right]$$

$$= pn - \sum_{m=k}^{\Delta} (m - k + 1)E[|A_m|] - \sum_{m=k+1}^{\Delta} (m - k)E[|B_m|].$$

Let us denote the vertices of $G$ by $v_1, v_2, ..., v_n$ and the corresponding vertex degrees by $d_1, d_2, ..., d_n$. We will need the following lemma:

**Lemma 2.** If $p = \left(\frac{1}{\tilde{c}_{k+1}(1+k)}\right)^{1/k}$, then, for any vertex $v_i \in V(G)$,

$$\left(\frac{d_i}{m}\right) (1 - p)^{d_i - m} \leq \left(\frac{\Delta}{m}\right) (1 - p)^{\Delta - m}, \quad m \geq k. \quad (3)$$

**Proof.** The inequality (3) holds if $d_i = \Delta$. It is also true if $d_i < m$ because in this case $\left(\frac{d_i}{m}\right) = 0$. Thus, we may assume that

$$m \leq d_i < \Delta.$$

Now, it is easy to see that inequality (3) is equivalent to the following:

$$(1 - p)^{\Delta - d_i} \geq \left(\frac{d_i}{m}\right) / \left(\frac{\Delta}{m}\right) = \frac{(\Delta - m)!/(d_i - m)!}{\Delta!/d_i!} = \prod_{i=0}^{\Delta-d_i-1} \frac{\Delta - m - i}{\Delta - i}. \quad (4)$$

Further, $\Delta \geq k$ implies $\frac{\Delta}{k} \leq \frac{\Delta - i}{k - i}$, where $0 \leq i \leq k - 1$. Taking into account that $\Delta > 0$, we obtain

$$\left(\frac{\Delta}{k}\right)^k \leq \prod_{i=0}^{k-1} \frac{\Delta - i}{k - i} = c_k < \tilde{c}_{k+1}(1+k)$$

or

$$\frac{1}{\tilde{c}_{k+1}(1+k)} < \left(\frac{k}{\Delta}\right)^k.$$

Thus,

$$p^k < \left(\frac{k}{\Delta}\right)^k \quad \text{or} \quad p < \frac{k}{\Delta} \leq \frac{m}{\Delta}.$$
We have $p < \frac{m}{\Delta}$, which is equivalent to $1 - p > \frac{\Delta - m}{\Delta}$. Therefore,
\[
(1 - p)^{\Delta - d_i} > \left( \frac{\Delta - m}{\Delta} \right)^{\Delta - d_i} \geq \prod_{i=0}^{\Delta - d_i - 1} \frac{\Delta - m - i}{\Delta - i},
\]
as required in (4).

Now we go on with the proof of Theorem 1. By Lemma 2,
\[
E[|A_m|] = \sum_{i=1}^{n} P[v_i \in A_m]
= \sum_{i=1}^{n} p \left( \frac{d_i}{m} \right) p^m(1 - p)^{d_i - m}
\leq p^{m+1} \sum_{i=1}^{n} \left( \frac{\Delta}{m} \right) (1 - p)^{\Delta - m}
= p^{m+1}(1 - p)^{\Delta - m} c_m n,
\]
where $p \left( \frac{d_i}{m} \right) p^m(1 - p)^{d_i - m}$ is the probability of having vertex $v_i$, $i = 1, \ldots, n$, in the set $A_m$, $m = k, \ldots, \Delta$. Again, by Lemma 2,
\[
E[|B_m|] = \sum_{i=1}^{n} P[v_i \in B_m]
= \sum_{i=1}^{n} (1 - p) \left( \frac{d_i}{m} \right) p^m(1 - p)^{d_i - m}
\leq p^m \sum_{i=1}^{n} \left( \frac{\Delta}{m} \right) (1 - p)^{\Delta - m + 1}
= p^m(1 - p)^{\Delta - m + 1} c_m n,
\]
where $(1 - p) \left( \frac{d_i}{m} \right) p^m(1 - p)^{d_i - m}$ is the probability of having vertex $v_i$, $i = 1, \ldots, n$, in the set $B_m$, $m = k + 1, \ldots, \Delta$.

Taking into account that $c_{\Delta+1} = \left( \frac{\Delta}{\Delta + 1} \right) = 0$, we obtain
\[
E[|X|] \geq pm - \sum_{m=k}^{\Delta} (m - k + 1)p^{m+1}(1 - p)^{\Delta - m} c_m n - \sum_{m=k+1}^{\Delta+1} (m - k)p^m(1 - p)^{\Delta - m + 1} c_m n
= pm - \sum_{m=0}^{\Delta-k} (m + 1)p^{m+k+1}(1 - p)^{\Delta - m - k} c_{m+k} n
- \sum_{m=0}^{\Delta-k} (m + 1)p^{m+k+1}(1 - p)^{\Delta - m - k} c_{m+k+1} n
\]

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\[ pn - \sum_{m=0}^{\Delta-k} (m+1)p^{m+k+1}(1-p)^{\Delta-m-k}n \left( c_{m+k} + c_{m+k+1} \right) \]
\[ = pn - p^{k+1}n \sum_{m=0}^{\Delta-k} (m+1)\tilde{c}_{m+k+1}p^m(1-p)^{\Delta-k-m}. \]

Furthermore,

\[
(m+1)\tilde{c}_{m+k+1} = \binom{\Delta-k}{m} \frac{(m+1)!(\Delta+1)!}{(m+k+1)!(\Delta-k)!} \leq \binom{\Delta-k}{m} \frac{(\Delta+1)!}{(k+1)!(\Delta-k)!} = \binom{\Delta-k}{m} \tilde{c}_{k+1}.
\]

We obtain, by the binomial theorem,

\[
E[|X|] \geq pn - p^{k+1}n \sum_{m=0}^{\Delta-k} \binom{\Delta-k}{m} \tilde{c}_{k+1}p^m(1-p)^{\Delta-k-m} \\
= pn - p^{k+1}n \tilde{c}_{k+1} \\
= pn(1-p^k \tilde{c}_{k+1}) \\
= \frac{kn}{\tilde{c}_{k+1}^{1/k} (1+k)^{1+1/k}}.
\]

Since the expectation is an average value, there exists a particular \( k \)-limited packing of size at least \( \frac{kn}{\tilde{c}_{k+1}^{1/k} (1+k)^{1+1/k}} \), as required. The proof of the theorem is complete. \( \square \)

The lower bound of Theorem 1 can be written in a simpler but weaker form as follows:

**Corollary 3.** For any graph \( G \) of order \( n \),

\[ L_k(G) > \frac{kn}{e(1+\Delta)^{1+1/k}}. \]

**Proof.** It is not difficult to see that

\[ \tilde{c}_{k+1} \leq \frac{(\Delta+1)^{k+1}}{(k+1)!} \]

and, using Stirling’s formula,

\[ (k!)^{1/k} > \left( \sqrt{2\pi k} \left( \frac{k}{e} \right)^k \right)^{1/k} = \frac{2^{k/2}}{\sqrt{2\pi k}} \frac{k^k}{e}. \]

By Theorem 1,

\[ L_k(G) \geq \frac{kn ((k+1)!)^{1/k}}{(\Delta+1)^{1+1/k} (1+k)^{1+1/k}} \geq \frac{kn}{e(1+\Delta)^{1+1/k}} \times \frac{2^{k/2}}{\sqrt{2\pi k}} \frac{k}{1+k} > \frac{kn}{e(1+\Delta)^{1+1/k}}. \]

6
Note that \( \frac{2\sqrt{2\pi k}}{k^{1+1/k}} > 1 \). The last inequality is obviously true for \( k = 1 \), while for \( k \geq 2 \) it can be rewritten in the equivalent form: \( 2\pi k > (1 + 1/k)^{2k} = e^2 - o(1) \).

In the case \( k = 1 \), Theorem 1 gives the following lower bound for the 2-packing (1-limited packing) number:

**Corollary 4.** For any graph \( G \) of order \( n \) with \( \Delta \geq 1 \),

\[
\rho(G) = L_1(G) \geq \frac{n}{2\Delta(\Delta + 1)}.
\]  

Let \( \delta = \delta(G) \) denote the minimum vertex degree in a graph \( G \). The lower bound of Corollary 4 can be improved as follows:

**Theorem 5.** For any graph \( G \) of order \( n \),

\[
\rho(G) = L_1(G) \geq \frac{n + \Delta(\Delta - \delta)}{\Delta^2 + 1} \geq \frac{n}{\Delta^2 + 1}.
\]  

**Proof.** Choose any vertex \( v \in V(G) \) of the minimum degree \( \delta \) in \( G \). Then add \( v \) to a set \( X \) and remove vertices of \( N[N[v]] \) from the graph to obtain \( G' = G - N[N[v]] \), where \( N[N[v]] = \{ w : w \in N[u] \text{ for some } u \in N[v] \} \) is the so-called second closed neighbourhood of \( v \) in \( G \). Recursively apply the same procedure to the remaining graph \( G' \) until it is empty. It is not difficult to see that \( X \) is a 1-limited packing (distance 2-packing) of size at least \( \left\lceil \frac{n + \Delta(\Delta - \delta)}{\Delta^2 + 1} \right\rceil \): we remove at most \( 1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2 \) vertices at each iteration, but at most \( 1 + \delta + \delta(\Delta - 1) = 1 + \delta \Delta \) vertices at the first iteration, and \( (1 + \Delta^2) - (1 + \delta \Delta) = \Delta(\Delta - \delta) \).

The proof of Theorem 5 provides a greedy algorithm to find a distance 2-packing (1-limited packing) satisfying bound (6). We explain later in Section 4 why the lower bound of Theorem 5 is as good as lower bound (5) of Corollary 4 for almost all graphs.

**3. Randomized algorithm**

A pseudocode presented in Algorithm 1 explicitly describes a randomized algorithm to find a \( k \)-limited packing set, whose size satisfies bound (1) with a positive probability. Notice that Algorithm 1 constructs a (preliminary) \( k \)-limited packing \( X' \) by recursively removing unwanted vertices from a randomly generated set \( A \). This is different from the probabilistic construction used in the proof of Theorem 1. The recursive removal of vertices from the set \( A \) may be more effective and efficient, especially if one tries to remove overall as few vertices as possible from \( A \) by maximizing intersections of the sets \( A'_m \) (\( m = k, \ldots, \Delta \)) and \( B'_m \) (\( m = k+1, \ldots, \Delta \)).

At the final stage, Algorithm 1 does a (greedy) extension of the preliminary \( k \)-limited packing \( X' \) derived from the randomly generated set \( A \). Our experimental tests with randomly generated problem instances show the following: although
the randomized part of Algorithm 1 may eventually return a preliminary \(k\)-limited packing set slightly smaller than lower bound (1), the extension of this set to a maximal \(k\)-limited packing always satisfies (1). This is of no surprise, because the expectation of the size of randomly formed set \(A\) in Algorithm 1 is \(E[|A|] = pn\), where \(p = \left( \frac{\Delta}{k} \right) (\Delta + 1)^{-1/k}\), while the expression for lower bound in (1) yields a smaller value:

\[
kn \frac{c_{k+1}^{1/k} (1+k)^{1+1/k}}{c_{k+1} (1+k)^{1+1/k}} = \frac{k}{k+1} \times \frac{m}{E[|A|]} < E[|A|].
\]

From the experimental tests, an initially formed set \(A\) may contain only few redundant vertices to be removed to obtain the preliminary \(k\)-limited packing \(X'\). As a result, the preliminary \(k\)-limited packing \(X'\) in many cases satisfies lower bound (1), and the extension of \(X'\) to a maximal \(k\)-limited packing \(X\) seems to always satisfy (1). In our view, since the problem is \(NP\)-hard, Algorithm 1 constitutes a simple efficient approach to tackle the problem in practice and, hopefully, can be useful to solve some hard instances of the problem.

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**Algorithm 1**: Randomized \(k\)-limited packing

**Input**: Graph \(G\) and integer \(k\), \(1 \leq k \leq \Delta\).

**Output**: \(k\)-Limited packing \(X\) in \(G\).

begin

\[
\text{Compute } p = \left( \frac{\Delta}{c_{k+1} (1+k)^{1+1/k}} \right)^{1/k};
\]

Initialize \(A = \emptyset\);

/* Form a set \(A \subseteq V(G)\) */

foreach vertex \(v \in V(G)\) do

| with the probability \(p\), decide whether \(v \in A\) or \(v \notin A\);

end /* Recursively remove redundant vertices from \(A\) */

foreach vertex \(v \in V(G)\) do

| Compute \(r = |N(v) \cap A|\);
| if \(v \in A\) and \(r \geq k\) then
| | remove any \(r - k + 1\) vertices of \(N(v) \cap A\) from \(A\);
| end
| if \(v \notin A\) and \(r > k\) then
| | remove any \(r - k\) vertices of \(N(v) \cap A\) from \(A\);
| end

end

Put \(X' = A\); /* \(X'\) is a \(k\)-limited packing */

Extend \(X'\) to a maximal \(k\)-limited packing \(X\);

return \(X\);

end

Algorithm 1 can be implemented to run in \(O(n^2)\) time. To compute the probability \(p = \left( \frac{\Delta}{k} \right) (\Delta + 1)^{-1/k}\), the binomial coefficient \(\binom{\Delta}{k}\) can be computed
by using the dynamic programming and Pascal’s triangle in $O(k\Delta) = O(\Delta^2)$ time using $O(k) = O(\Delta)$ memory. The maximum vertex degree $\Delta$ of $G$ can be computed in $O(m)$ time, where $m$ is the number of edges in $G$. Then $p$ can be computed in $O(m + \Delta^2) = O(n^2)$ steps. It takes $O(n)$ time to find the initial set $A$. Computing the intersection numbers $r = |N(v) \cap A|$ and removing unwanted vertices of $N(v) \cap A$’s from $A$ can be done in $O(n + m)$ steps. Finally, checking whether $X'$ is maximal or extending $X'$ to a maximal $k$-limited packing $X$ can be done in $O(n + m)$ time: try to add vertices of $V(G) - X'$ to $X'$ recursively one by one, and check whether the addition of a new vertex $v \in V(G) - X'$ to $X'$ violates the conditions of a $k$-limited packing for $v$ or at least one of its neighbours in $G$ with respect to $X' \cup \{v\}$. Thus, overall Algorithm 1 takes $O(n^2)$ time, and, since $m = O(n^2)$ in general, it is linear in the graph size $(m + n)$ when $m = \theta(n^2)$.

Also, this randomized algorithm for finding $k$-limited packings in a graph $G$ can be implemented in parallel or as a local distributed algorithm. As explained in [5], this kind of algorithms are especially important, e.g. in the context of ad hoc and wireless sensor networks. We hope that this approach can be also extended to design self-stabilizing or on-line algorithms for $k$-limited packings. For example, a self-stabilizing algorithm searching for maximal 2-packings in a distributed network system is presented in [12]. Notice that self-stabilizing algorithms are distributed and fault-tolerant, and use the fact that each node has only a local view/knowledge of the distributed network system. This provides another motivation for efficient distributed search and algorithms to find $k$-limited packings in graphs and networks.

4. Sharpness of the lower bounds

We now show that the lower bound of Theorem 1 is asymptotically best possible for some values of $k$. The bound of Theorem 1 can be rewritten in the following form for $\Delta \geq k$:

$$L_k(G) \geq \frac{kn}{(k + 1)\sqrt[k]{\binom{\Delta}{k}}(\Delta + 1)}.$$

Combining this bound with the upper bound of Lemma 8 from [6], we obtain that for any connected graph $G$ of order $n$ with minimum degree $\delta(G) \geq k$,

$$\frac{1}{\sqrt[k]{\binom{\Delta}{k}}(\Delta + 1)} \times \frac{k}{k + 1}n \leq L_k(G) \leq \frac{k}{k + 1}n. \quad (7)$$

Notice that the upper bound in the inequality (7) is sharp (see [6]), so these bounds provide an interval of values for $L_k(G)$ in terms of $k$ and $\Delta$ when $k \leq \delta$. For regular graphs, $\delta = \Delta$, and, when $k = \Delta$, we have

$$\frac{1}{\sqrt[k]{\binom{\Delta}{k}}(\Delta + 1)} = \frac{1}{(k + 1)^{1/k}} \rightarrow 1 \quad \text{as} \quad k \rightarrow \infty.$$
Therefore, the bound of Theorem 1 is asymptotically sharp for regular connected graphs in the case $k = \Delta$. In other words, there are graphs whose $k$-limited packing number is arbitrarily close to the bound of Theorem 1. Thus, the following result holds:

**Theorem 6.** When $n$ is large, there exist graphs $G$ such that

$$L_k(G) \leq \frac{kn}{\frac{1}{c_{k+1}^{1/k}} (1 + k)^{1+1/k}} (1 + o(1)).$$  \hspace{1cm} (8)

As shown above, the graphs satisfying Theorem 6 contain regular connected ones for $k = \Delta$. This class of graphs can be extended, because it is possible to prove that the bound of Theorem 1 is asymptotically sharp for connected graphs with $k = \Delta(1 - o(1))$, $\delta(G) \geq k$.

Notice that, for regular graphs, the condition $k = \Delta$ and Lemma 5 from [6] imply $L_k(G) = n - \gamma(G)$. Then the classical upper bound (9) for $\gamma(G)$ gives a weaker lower bound for $L_k(G)$ than Theorem 1.

As shown in Theorem 5, in contrast to the situation for relatively ‘large’ values of $k$, bound (1) of Theorem 1 (see Corollary 4) can be improved for distance 2-packings (1-limited packings), i.e. when $k = 1$. However, this improvement is irrelevant for almost all graphs. A 1-limited packing set $X$ in $G$ has a very strong property that any two vertices in $X$ are at distance at least 3 in $G$. It is well known that almost every graph has diameter equal to 2 (e.g., see [10]). Therefore, $\rho(G) = L_1(G) = 1$ for almost all graphs. Thus, in the case $k = 1$, Theorem 1 yields a lower bound of 1 for almost all graphs and is as good as Theorem 5. Notice that the bound of Theorem 5 is sharp, for example, for any number of disjoint copies of the Petersen graph. In the other cases, when $G$ has a diameter larger than 2, one is encouraged to use the greedy algorithm and lower bound (6) provided by Theorem 5, because it improves bound (5) of Corollary 4 by a factor of $2 + o(1)$.

5. Upper bounds

As mentioned earlier, $\rho(G) = L_1(G) \leq \gamma(G)$. In [6], the authors provide several upper bounds for $L_k(G)$, e.g. $L_k(G) \leq k\gamma(G)$ for any graph $G$. Using the well-known bound (see e.g. [1])

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n,$$  \hspace{1cm} (9)

we obtain

$$L_k(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} kn.$$  \hspace{1cm} (10)

Even though this bound does not work well when $k$ is ‘close’ to $\delta$, it is very reasonable for small values of $k$.

We now prove an upper bound for the $k$-limited packing number in terms of the $k$-tuple domination number. A set $X$ is called a $k$-tuple dominating set of $G$ if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a $k$-tuple dominating set of $G$ is the $k$-tuple domination number $\gamma_{\times k}(G)$. The $k$-tuple domination number is only defined for graphs with $\delta \geq k - 1$. 
Theorem 7. For any graph $G$ of order $n$ with $\delta \geq k - 1$,
\[
L_k(G) \leq \gamma_{\times k}(G).
\] (11)

Proof. We prove inequality (11) by contradiction. Let $X$ be a maximum $k$-limited packing in $G$ of size $L_k(G)$, and let $Y$ be a minimum $k$-tuple dominating set in $G$ of size $\gamma_{\times k}(G)$. We denote $B = X \cap Y$, i.e. $X = A \cup B$ and $Y = B \cup C$, where $A$ and $C$ are disjoint. Assume to the contrary that $L_k(G) > \gamma_{\times k}(G)$, thus $|A| > |C|$.

Since $Y$ is $k$-tuple dominating set, each vertex of $A$ is adjacent to at least $k$ vertices of $Y$. Hence the number of edges between $A$ and $B \cup C$ is as follows:
\[
e(A, B \cup C) \geq k|A|.
\]
Now, every vertex of $C$ is adjacent to at most $k$ vertices of $X$, because $X$ is a $k$-limited packing set. Therefore, the number of edges between $C$ and $A \cup B$ satisfies
\[
e(C, A \cup B) \leq k|C|.
\]
We obtain
\[
e(C, A \cup B) \leq k|C| < k|A| \leq e(A, B \cup C),
\]
i.e. $e(C, A \cup B) < e(A, B \cup C)$. By eliminating the edges between $A$ and $C$, we conclude that
\[
e(C, B) < e(A, B).
\]

Now, let us consider an arbitrary vertex $b \in B$ and denote $s = |N(b) \cap A|$. Since $X = A \cup B$ is a $k$-limited packing set, we obtain $|N(b) \cap X| \leq k - 1$, and hence $|N(b) \cap B| \leq k - s - 1$. On the other hand, $Y = B \cup C$ is $k$-tuple dominating set, so $|N(b) \cap Y| \geq k - 1$. Therefore, $|N(b) \cap C| \geq s$. Thus, $|N(b) \cap C| \geq |N(b) \cap A|$ for any vertex $b \in B$. We obtain
\[
e(C, B) \geq e(A, B),
\]
a contradiction. We conclude that $L_k(G) \leq \gamma_{\times k}(G)$. \hfill \qed

Notice that it is possible to have $k = \Delta + 1$ in the statement of Theorem 7, which is not covered by Theorem 1. Then $\delta = \Delta$, which implies the graph is regular. However, $L_k(G) = \gamma_{\times k}(G) = n$ for $k = \delta + 1 = \Delta + 1$. In non-regular graphs, $\delta + 1 \leq \Delta$, and $k \leq \Delta$ to satisfy the conditions of Theorem 1 as well.

For $t \leq \delta$, we define
\[
\delta' = \delta - k + 1 \quad \text{and} \quad \overline{b}_t = \overline{b}_t(G) = \left(\frac{\delta + 1}{t}\right).
\]
Using the upper bound for the $k$-tuple domination number from [5], we obtain:

Corollary 8. For any graph $G$ with $\delta \geq k$,
\[
L_k(G) \leq \left(1 - \frac{\delta'}{\overline{b}_k^{-1}} \frac{\delta'}{1 + \delta'} \right)n. \tag{12}
\]
In some cases, Theorem 1 and Corollary 8 simultaneously provide good bounds for the $k$-limited packing number. For example, for a 40-regular graph $G$:
\[
0.312n < L_{25}(G) < 0.843n.
\]
Acknowledgement

The authors are grateful to the anonymous referee for helpful comments and suggestions.

References


