The Domination Parameters of Cubic Graphs

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Abstract

Let ir(G), γ(G), i(G), β0(G), Γ(G) and IR(G) be the irredundance number, the domination number, the independent domination number, the independence number, the upper domination number and the upper irredundance number of a graph G, respectively. In this paper we show that for any nonnegative integers k1, k2, k3, k4, k5 there exists a cubic graph G satisfying the following conditions: γ(G) − ir(G) ≥ k1, i(G) − γ(G) ≥ k2, β0(G) − i(G) ≥ k3, Γ(G) − β0(G) ≥ k4, and IR(G) − Γ(G) ≥ k5. This result settles a problem posed in [9].

1 Introduction and Main Result

All graphs will be finite and undirected without multiple edges. If G is a graph, V(G) denotes the set, and |G| the number, of vertices in G. Let N(x) denote the neighborhood of a vertex x, and let ⟨X⟩ denote the subgraph of G induced by X ⊆ V(G). Also let N(X) = ∪x∈X N(x) and N[X] = N(X) ∪ X.

A set I ⊆ V(G) is called independent if no two vertices of I are adjacent. A set X is called a dominating set if N[X] = V(G). An independent dominating set is a vertex subset that is both independent and dominating, or equivalently, is maximal independent.

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The independence number $\beta_0(G)$ is the maximum cardinality of a (maximal) independent set of $G$, and the independent domination number $i(G)$ is the minimum cardinality taken over all maximal independent sets of $G$. The domination number $\gamma(G)$ is the minimum cardinality of a (minimal) dominating set of $G$, and the upper domination number $\Gamma(G)$ is the maximum cardinality taken over all minimal dominating sets of $G$. For $x \in X$, the set

$$PN(x, X) = PN(x) = N[x] - N[X - \{x\}]$$

is called the private neighborhood of $x$. If $PN(x, X) = \emptyset$, then $x$ is said to be redundant in $X$. A set $X$ containing no redundant vertex is called irredundant. The irredundance number $ir(G)$ is the minimum cardinality taken over all maximal irredundant sets of $G$, and the upper irredundance number $IR(G)$ is the maximum cardinality of a (maximal) irredundant set of $G$. An $ir$-set $X$ of $G$ is a maximal irredundant set of cardinality $ir(G)$. A $\gamma$-set, an $i$-set, a $\beta_0$-set, a $\Gamma$-set and an $IR$-set are defined analogously.

The following relationship among the parameters under consideration is well-known [2, 3]:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

The above and related parameters for regular graphs were investigated by many authors [1], [4]–[17]. For example, Cockayne and Mynhardt [4] and independently Rautenbach [15] disproved the Henning-Slater conjecture [12] that $\Gamma(G) = IR(G)$ for any cubic graph $G$, while the Barefoot-Harary-Jones conjecture on the difference between the domination and independent domination numbers of cubic graphs was investigated in [5, 13, 14, 17].

In this paper, we deal with the next problem:

**Problem 1** ([9]) Does there exist a cubic graph for which $ir < \gamma < i < \beta_0 < \Gamma < IR$?

We define the graph $W_k$ ($k \geq 0$) as follows. Take a disjunct union of the graphs

$$F_1 \cong F_2 \cong \ldots \cong F_{2k+8}, G_1 \cong G_2 \cong \ldots \cong G_{2k+6}, H_1 \cong H_2 \cong \ldots \cong H_{3k+6},$$

where $F_i, G_i$ and $H_i$ are shown in Figure 1, and add the edges

$$\{f_i', f_{i+1} : 1 \leq i \leq 2k + 7\}, f_{2k+8}'g_1,$$

$$\{g_i', g_{i+1} : 1 \leq i \leq 2k + 5\}, g_{2k+6}'h_1,$$

$$\{h_i', h_{i+1} : 1 \leq i \leq 3k + 5\}, h_{3k+6}'f_1.$$  

**Theorem 1** For any nonnegative integers $k_1, k_2, k_3, k_4, k_5$ there exists an integer $k$ such that the cubic graph $W_k$ satisfies the following conditions: $\gamma(W_k) - ir(W_k) \geq k_1$, $i(W_k)$, $\gamma(W_k) \geq k_2$, $\beta_0(W_k) - i(W_k) \geq k_3$, $\Gamma(W_k) - \beta_0(W_k) \geq k_4$, and $IR(W_k) - \Gamma(W_k) \geq k_5$.

It follows from Lemmas 1–5 of Section 2 that the graph $W_0$ has the property

$$ir < \gamma < i < \beta_0 < \Gamma < IR,$$

thus solving Problem 1.

We conclude this section with the next conjecture.

**Conjecture 1** For any integers $k_1, k_2, k_3, k_4, k_5$ there exists a 3-connected cubic graph $G$ satisfying the following conditions: $\gamma(G) - ir(G) \geq k_1$, $i(G) - \gamma(G) \geq k_2$, $\beta_0(G) - i(G) \geq k_3$, $\Gamma(G) - \beta_0(G) \geq k_4$, and $IR(G) - \Gamma(G) \geq k_5$. 

Figure 1. Graphs $F_i$, $G_i$, and $H_i$.

2 Proof of Theorem 1

The proof of Theorem 1 is based on five lemmas. Let us denote by $F, G$ and $H$ the graphs induced by the sets $\bigcup_{i=1}^{2k+8} V(F_i)$, $\bigcup_{i=1}^{2k+6} V(G_i)$, and $\bigcup_{i=1}^{3k+6} V(H_i)$, respectively.

Lemma 1 $\gamma(W_k) - \gamma(W_k) \geq k + 1$.

Proof: Let $D$ denote a $\gamma$-set of $W_k$. It is straightforward to check that $|D \cap V(G_i)| = 4$ whenever both $g_i$ and $g'_i$ are dominated by $D - V(G_i)$, and $|D \cap V(G_i)| = 5$ otherwise. Moreover, if $|D \cap V(G_i)| = 4$, then $g_i, g'_i \not\in D$. Thus, the number of components $G_i$ satisfying $|D \cap V(G_i)| = 4$ is at most $k + 3$. We obtain

$$|D \cap V(G)| \geq 4(k + 3) + 5(k + 3) = 9k + 27.$$
Consider the set $J = (D - V(G)) \cup R$, where
\[
R = \{N(g_i) \cap V(G_i), N(g'_i) \cap V(G_i) : 1 \leq i \leq 2k + 6\}.
\]
We have
\[
|R| = 8k + 24.
\]
Let us construct a maximal irredundant set of $W_k$. We first put $J' = J$. Further, if $N[h_1] \cap J = \emptyset$, then we put $g'_2k+6 \in J'$. If $N[f'_2k+8] \cap J = \emptyset$, then we put $g_1 \in J'$. If $h_1 \in D$ and $PN(h_1, D) = g'_2k+6$, then we put $h_1 \not\in J'$. Finally, if $f'_2k+8 \in D$ and $PN(f'_2k+8, D) = g_1$, then we put $f'_2k+8 \not\in J'$. It is easy to see that the set $J'$ is a maximal irredundant set, and $|J'| \leq |J| + 2$. We obtain
\[
\gamma(W_k) - ir(W_k) \geq |D| - |J'| \geq |D| - |J| - 2 = |D \cap V(G)| - |R| - 2 \geq k + 1.
\]

\textbf{Lemma 2} $i(W_k) - \gamma(W_k) \geq k + 1$.

\textbf{Proof:} We denote by $I$ an $i$-set of $W_k$.

\textbf{Claim 1} We have $|I \cap V(H_i)| = 3$ or $4$ for any $i$, $1 \leq i \leq 3k+6$. Moreover, $|I \cap V(H_i)| = 3$ if and only if either $h_i$ or $h'_i$ is dominated by $I - V(H_i)$, and additionally $h_i, h'_i \not\in I$.

\textbf{Proof:} Assume that $h_i, h'_i \in I$ for some $i$, $1 \leq i \leq 3k + 6$. We obtain $|I \cap V(H_i)| = 4$. Suppose now that exactly one vertex from $h_i, h'_i$ belongs to $I$, say $h_i \in I$ and $h'_i \not\in I$. If $b_i, c_i \not\in I$, then these vertices cannot be dominated by an independent set, a contradiction. Therefore, without loss of generality, $b_i \in I$ and $c_i \not\in I$. Hence $a'_i \in I$, and either $c'_i \in I$ or $d'_i \in I$. We have $|I \cap V(H_i)| = 4$. Consider the case $h_i, h'_i \not\in I$. Since $I \cap \{b_i, b'_i, c_i, c'_i\} \neq \emptyset$, we may assume without loss of generality that $b_i \in I$ and hence $a'_i \in I$. If $c'_i \in I$, then $d_i \in I$ and $|I \cap V(H_i)| = 4$. If $c'_i \not\in I$, then $d'_i \in I$ and $|I \cap V(H_i)| = 3$.

By Claim 1, the number of components $H_i$ satisfying $|I \cap V(H_i)| = 3$ is at most $2k + 4$. Therefore,
\[
|I \cap V(H)| \geq 10k + 20.
\]
Let us consider the set $D = \{h'_{3k+6}, b'_i, c'_i : i = 1, 2, ..., 3k + 6\}$. It is evident that the set $J = (I - V(H)) \cup D$ is a dominating set of $W_k$ and
\[
i(W_k) - \gamma(W_k) \geq |I| - |J| = |I \cap V(H)| - |D| \geq 10k + 20 - 9k - 19 = k + 1.
\]

Now we estimate the difference between the independence and independent domination numbers of $W_k$.

\textbf{Lemma 3} $\beta_0(W_k) - i(W_k) \geq 2k + 4$. 

Proof: It is easy to construct a maximal independent set $I$ of $W_k$ such that $|I \cap V(F_i)| = 6$, $|I \cap V(G_i)| = 6$, and $|I \cap V(H_i)| = 4$. We define the set $R \subset V(H)$ as follows. For each $i \in \{1, 2, \ldots, 3k + 6\}$, we put $a_i, d_i, b'_i \in R$ if $i = 1 \pmod 3$, $b_i, h'_i, b'_i, c_i \in R$ if $i = 2 \pmod 3$, and $a'_i, b_i, d'_i \in R$ if $i = 0 \pmod 3$. Now, the set $J = (I - V(H)) \cup R$ is an independent dominating set and hence $i(W_k) \leq |J|$. We obtain

$$\beta_0(W_k) - i(W_k) \geq |I| - |J| = |I \cap V(H)| - |R| = 12k + 24 - 10k - 20 = 2k + 4.$$ 

Lemma 4 $\Gamma(W_k) - \beta_0(W_k) \geq 3k + 5$.

Proof: We can split $V(F_i)$ into three cycles $C_3$ and one $C_7$, $V(G_i)$ into two cycles $C_5$ and two cycles $C_3$, and $V(H_i)$ into two cycles $C_5$. Therefore,

$$\beta_0(W_k) \leq 6(2k + 8) + 6(2k + 6) + 4(3k + 6) = 36k + 108.$$ 

It is easy to construct a maximal independent set $I$ of $W_k$ such that $|I \cap V(F_i)| = 6$, $|I \cap V(G_i)| = 6$ and $g_{2k+6}' \in I$, and $|I \cap V(H_i)| = 4$. Thus, $|I| = 36k + 108$ and hence $\beta_0(W_k) = |I|$.

Consider the set $S = \{h'_i, a'_i, b'_i, c'_i, d'_i : 1 \leq i \leq 3k + 6\} - \{h_{3k+6}'\}$. It is evident that $R = (I - V(H)) \cup S$ is a minimal dominating set. We have

$$\Gamma(W_k) - \beta_0(W_k) \geq |R| - |I| = |S| - |I \cap V(H)| = 15k + 29 - 12k - 24 = 3k + 5.$$ 

Denote by $D$ a $\Gamma$-set of $W_k$.

Proposition 1 $|D \cap V(F)| \leq 13k + 53$.

Proof: Let us label the vertices of $F_i$ as shown in Figure 2, and put $X = \{x, a, b, h, i, j\}$.

![Figure 2](image_url)
Claim 2 \(|D \cap X| = 2\). Moreover, \(f, e, m \notin D\) if \(c, d \in D\) and at least one of the vertices \(k, k', p\) belongs to \(D\).

Proof: Since \(\{a, b, h, i\}\) is dominated by \(D \cap X\) and at least two vertices are required to dominate it, \(|D \cap X| \geq 2\). Suppose \(|D \cap X| \geq 3\). If \(|D \cap \{a, b, x\}| \geq 2\), then without loss of generality \(a \in D\) and \(PN(a) = \{h\}\). Thus \(h, i, j \notin D\), so \(x \in D\) and \(PN(x) = \{d\}\), whence \(c \notin D\). Hence \(j\) is not dominated, a contradiction. A similar contradiction shows that \(|D \cap \{h, i, j\}| \geq 2\) is impossible. Therefore \(|D \cap X| = 2\).

Suppose that \(c, d \in D\) and at least one of the vertices \(k, k', p\) belongs to \(D\). We have \(\{x\} = PN(d)\) and hence \(a, b, x \notin D\). Therefore \(h, i \in D\) and \(PN(c) = \{e\}\). Hence \(m, e, f \notin D\).

We define 16 types for the component \(F_i\) as follows:

- \(F_i\) has type A1 if \(k', g' \in D\) and \(k \in PN(k'), g \in PN(g')\);
- \(F_i\) has type A2 if \(k', g' \in D\) and \(k \notin PN(k'), g \in PN(g')\);
- \(F_i\) has type A3 if \(k', g' \in D\) and \(k \in PN(k'), g \notin PN(g')\);
- \(F_i\) has type A4 if \(k', g' \in D\) and \(k \notin PN(k'), g \notin PN(g')\);
- \(F_i\) has type B1 if \(k' \in D, g' \notin D\) and \(k \in PN(k'), g' \in N(D - V(F_i))\);
- \(F_i\) has type B2 if \(k' \in D, g' \notin D\) and \(k \notin PN(k'), g' \in N(D - V(F_i))\);
- \(F_i\) has type B3 if \(k' \in D, g' \notin D\) and \(k \in PN(k'), g' \in PN(g)\);
- \(F_i\) has type B4 if \(k' \in D, g' \notin D\) and \(k \notin PN(k'), g' \in PN(g)\);
- \(F_i\) has type C1 if \(k' \notin D, g' \in D\) and \(k' \in N(D - V(F_i)), g \in PN(g')\);
- \(F_i\) has type C2 if \(k' \notin D, g' \in D\) and \(k' \in N(D - V(F_i)), g \notin PN(g)\);
- \(F_i\) has type C3 if \(k' \notin D, g' \in D\) and \(k' \in PN(k), g \notin PN(g')\);
- \(F_i\) has type C4 if \(k' \notin D, g' \in D\) and \(k' \in PN(k), g \in PN(g')\);
- \(F_i\) has type D1 if \(k', g' \notin D\) and \(k' \in N(D - V(F_i)), g' \in N(D - V(F_i))\);
- \(F_i\) has type D2 if \(k', g' \notin D\) and \(k' \in PN(k), g' \in N(D - V(F_i))\);
- \(F_i\) has type D3 if \(k', g' \notin D\) and \(k' \in N(D - V(F_i)), g' \in PN(g)\);
- \(F_i\) has type D4 if \(k', g' \notin D\) and \(k' \in PN(k), g' \in PN(g)\).

Let us denote \(D_i = D \cap V(F_i)\).

Claim 3 We have

- \((a1)\) \(|D_i| = 5\) if \(F_i\) is of type A1;
- \((a2)\) \(|D_i| = 6\) if \(F_i\) is of type A2;
- \((a3)\) \(|D_i| = 5\) if \(F_i\) is of type A3;
- \((a4)\) \(|D_i| = 6\) if \(F_i\) is of type A4;
- \((b1)\) \(|D_i| = 5\) if \(F_i\) is of type B1;
- \((b2)\) \(|D_i| = 6\) if \(F_i\) is of type B2;
- \((b3)\) \(|D_i| = 5\) if \(F_i\) is of type B3;
- \((b4)\) \(|D_i| = 7\) if \(F_i\) is of type B4;
- \((c1)\) \(|D_i| = 6\) if \(F_i\) is of type C1;
- \((c2)\) \(|D_i| = 6\) if \(F_i\) is of type C2;
- \((c3)\) \(|D_i| = 7\) if \(F_i\) is of type C3;
- \((c4)\) \(|D_i| = 6\) if \(F_i\) is of type C4;
- \((d1)\) \(|D_i| = 6\) if \(F_i\) is of type D1;
- \((d2)\) \(|D_i| = 7\) if \(F_i\) is of type D2;
(d3) $|D_i| = 7$ if $F_i$ is of type D3;
(d4) $|D_i| = 8$ if $F_i$ is of type D4.

**Proof:** In what follows we will use the first part of Claim 2 without further reference.

(a1) Since $k \in PN(k')$ and $g \in PN(g')$, we have $d, k, p, g, f \notin D$. Also, $y \in D$, for otherwise $p$ is not dominated. Suppose that $e \in D$. We have $m, n \notin D$. Now we can use the vertex $c$ and two vertices of $X$ to construct $D_i$ such that $|D_i| = 5$. Assume that $e \notin D$. We obtain $n \in D$. It is easy to see that exactly one of the vertices $c, m$ belongs to $D$ and hence $|D_i| = 5$.

(a2) We have $p, g, f \notin D$. If $c \notin D$, then $|D_i - X| = 4$ and hence $|D_i| = 6$. Suppose that $c \in D$. If $d \in D$, then $m, e \notin D$ by Claim 2. Hence $n \in D$ and $|D_i| = 6$. If $d \notin D$, then again $|D_i| = 6$.

(a3) We have $d, k \notin D$. If $c \notin D$, then $|D_i - X| = 3$ and hence $|D_i| = 5$. Consider the case $c \in D$. If $y \in D$, then $|D_i| = 5$. If $y \notin D$, then $g \in D$, for otherwise $p$ is not dominated. To dominate $y$ we must take either $m$ or $n$ and hence $|D_i| = 5$.

(a4) Assume that $c, d \notin D$. It is not difficult to see that $|D_i - X| = 4$ and hence $|D_i| = 6$. Consider the case $c \notin D$ and $d \in D$. If $k \in D$, then $\{p\} = PN(k)$ and hence $g, p, y \notin D$. We have $|D_i| = 6$. If $k \notin D$, then one can easily check that again $|D_i| = 6$. The case $c \in D$ and $d \notin D$ is analogous. Finally, suppose that $c, d \in D$. By Claim 2, $e, f, m \notin D$. If $k \in D$, then $\{p\} = PN(k)$. Therefore, $y, p, g \notin D$, $n \in D$ and $|D_i| = 6$. If $k \notin D$, then exactly two vertices from $\{n, y, p, g\}$ belong to $D$ and $|D_i| = 6$.

(b1) We have $d, k \notin D$. Suppose that $c \notin D$. If $y \in D$, then $|D_i - X| = 3$ and hence $|D_i| = 5$. If $y \notin D$, then $g \in D$ to dominate $p$. Again, $|D_i - X| = 3$ and $|D_i| = 5$. Consider the case $c \in D$. If $y \in D$, then $f \in D$ or $g \in D$, for otherwise $g$ is not dominated. We have $|D_i| = 5$. If $y \notin D$, then $g \in D$, for otherwise $p$ is not dominated. Also, one of the vertices $m, n$ belongs to $D$ to dominate $y$. We obtain $|D_i| = 5$.

(b2) Suppose that $c, d \notin D$. It is not difficult to see that $|D_i - X| = 4$ and hence $|D_i| = 6$. Consider the case $|D \cap \{c, d\}| = 1$. If $k \in D$, then $\{p\} = PN(k)$ and hence $g, p, y \notin D$. We have $|D_i| = 6$. If $k \notin D$, then one can easily check that again $|D_i| = 6$. Finally, assume that $c, d \in D$. By Claim 2, $f, e, m \notin D$. If $k \in D$, then $PN(k) = \{p\}$ and hence $g, p, y \notin D$. Now $g$ is not dominated, a contradiction. If $k \notin D$, then $|D_i| = 6$.

(b3) We have $d, k \notin D$ and $g \in D$. Suppose that $c \notin D$. If $y \in D$, then $f, m \notin D$, $e \in D$ and hence $|D_i| = 5$. If $y \notin D$, then again $|D_i| = 5$. Consider the case $c \in D$. To dominate $y$, exactly one of the vertices $m, n, y$ belongs to $D$. Hence $|D_i| = 5$.

(b4) We have $g \in D$. Suppose that $c, d \notin D$. It is not difficult to see that $|D_i - X| = 5$ and hence $|D_i| = 7$. Consider the case $|D \cap \{c, d\}| = 1$. If $k \in D$, then $PN(k) = \emptyset$, a contradiction. Therefore, $k \notin D$. It is easy to see that $|D_i| = 7$. Finally, assume that $c, d \in D$. By Claim 2, $f, e, m \notin D$. If $k \in D$, then $PN(k) = \emptyset$, a contradiction. Therefore, $k \notin D$. We obtain $|D_i| = 6$. Since $D$ is a maximum minimal dominating set, we conclude that $|D_i| = 7$.

(c1) We have $f, g, p \notin D$. Suppose that $k \notin D$. We obtain $y \in D$ to dominate $p$, and $d \in D$ to dominate $k$. Therefore, $|D_i| = 6$. Consider the case $k \in D$. If $c, d \notin D$, then $|D_i| = 5$. If exactly one vertex from $\{c, d\}$ is present in $D$, then it is checked directly that $|D_i| = 6$. Finally, suppose that $c, d \in D$. By Claim 2, $e, m \notin D$. We have $|D_i| = 6$. Since $D$ is a maximum minimal dominating set, we conclude that $|D_i| = 6$.

(c2) Assume that $c, d \notin D$. It is not difficult to see that $|D_i - X| = 4$ and hence $|D_i| = 6$. Consider the case $c \notin D$ and $d \in D$. If $k \in D$, then $\{p\} = PN(k)$ and hence
If difficult to see that one can easily check that again. Again, for $D_3$, then at least one of the properties (i) and (ii) holds. If $k \notin D$, then one can easily check that again.

Case 1. $k \in D$. By Claim 2, $e, f, m \notin D$. Further, $\{p\} = PN(k)$. Therefore, $g, p, y \notin D$, $n \in D$ and $|D_i| = 6$. 

Case 2. $k \notin D$. Suppose that $p \in D$. By Claim 2, $e, f, m \notin D$. Also, $y \notin D$, for otherwise $PN(p) = \emptyset$. We obtain $n \in D$ and $|D_i| = 6$. Assume now that $p \notin D$. If $y \in D$, then $|D_i| = 6$. If $y \notin D$, then $g \in D$ to dominate $p$. Moreover, exactly one vertex from $\{m, n\}$ belongs to $D$. Thus, $|D_i| = 6$.

(c3) We have $k \in D$. Suppose that $c, d \in D$. By Claim 2, $f, e, m \notin D$. We see that $|D_i| = 7$. Consider the case $|D \cap \{c, d\}| = 1$. It is checked directly that $|D_i| = 7$. If $c \notin D$, then $|D_i| = 6$. Since $D$ is a maximum minimal dominating set, we conclude that $|D_i| = 7$.

(c4) We have $f, g, p \notin D$. Suppose that $c \notin D$. It is checked directly that $|D_i| = 6$. Consider the case $c \in D$. If $d \notin D$, then $|D_i| = 6$. If $d \in D$, then $e, m \notin D$ by Claim 2. Again, $|D_i| = 6$.

(d1) Assume that $c, d \notin D$. If $k \notin D$, then $p \in D$ and $|D_i| = 5$. If $k \in D$, then it is not difficult to see that $|D_i - X| = 4$ and hence $|D_i| = 6$. Consider the case $c \notin D$ and $d \in D$. If $k \in D$, then $\{p\} = PN(k)$ and hence $g, p, y \notin D$. We have $|D_i| = 6$. If $k \notin D$, then one can easily check that again $|D_i| = 6$. Consider the case $c \in D$ and $d \notin D$. If $k \notin D$, then $p \in D$ and $|D_i| = 6$. If $k \in D$, then $p \notin D$, for otherwise $PN(k) = \emptyset$. It is easy to see that $|D_i| = 6$. Finally, suppose that $c, d \in D$. By Claim 2, $e, f, m \notin D$. If $k \in D$, then $\{p\} = PN(k)$. Therefore, $y, p, g \notin D$, $n \in D$ and $|D_i| = 6$. If $k \notin D$, then exactly two vertices from $\{n, y, p, g\}$ belong to $D$ and $|D_i| = 6$. Since $D$ is a maximum minimal dominating set, we conclude that $|D_i| = 6$.

(d2) The proof is analogous to the case (c3).

(d3) We have $g \in D$. The only difference between this case and the case (b4) is that the vertex $k$ is dominated by $k'$ in the latter case. Hence, if $d \in D$ or $k \in D$, then we use the corresponding reasoning of the case (b4) and obtain $|D_i| = 7$. Suppose now that $d, k \notin D$. We have $p \in D$, for otherwise $k$ is not dominated. Obviously $c, e, f \in D$ and $|D_i| = 7$.

(d4) We have $k, g \in D$. Suppose that $c, d \notin D$. It is not difficult to see that $D_i - X = \{k, e, f, g, p\}$. Hence $|D_i| = 7$. If $|D \cap \{c, d\}| = 1$, then $|D_i| = 8$. Finally, assume that $c, d \in D$. By Claim 2, $f, e, m \notin D$ and hence $|D_i| = 7$. Since $D$ is a maximum minimal dominating set, we conclude that $|D_i| = 8$.

Claim 4 If $F_i$ ($2 \leq i \leq 2k + 7$) has type $D_4$, then both (i) and (ii) hold; if $F_i$ has type $D_4$ and $i = 2k + 8$, then (i) holds. Furthermore, if $F_i$ ($2 \leq i \leq 2k + 7$) is of type $B_4$, $C_3$, $D_2$ or $D_3$, then at least one of the properties (i) and (ii) holds.

(i) $F_{i-1}$ has type $A_1$, $A_2$, $C_1$ or $C_4$ and $|D_{i-1}| \leq 6$.

(ii) $F_{i+1}$ has type $A_1$, $A_3$, $B_1$ or $B_3$ and $|D_{i+1}| = 5$.

Proof: This follows immediately from the definition and Claim 3.

Let $F_i$ be a component of type $D_4$ for some $i \leq 2k + 7$. By Claim 3, $|D_i| = 8$. By Claim 4, $F_{i+1}$ has type $A_1$, $A_3$, $B_1$ or $B_3$ and $|D_{i+1}| = 5$. We denote by $m$ the number of such
pairs. These components contain exactly $13m$ vertices of $D$, and any other component $F_j$ with $j \leq 2k + 7$ has $|D_j| \leq 7$. Suppose that there exist three sequential components $F_i$, $F_{i+1}$, $F_{i+2}$ such that $|D_i| = |D_{i+1}| = |D_{i+2}| = 7$, i.e., they are of type B4, C3, D2 or D3 by Claim 3. Applying Claim 4 to $F_{i+1}$ we arrive at a contradiction. Consider two components $F_i, F_{i+1}$ of type B4, C3, D2 or D3 such that $i \leq 2k + 6$. We have $|D_i| = |D_{i+1}| = 7$. Applying Claim 4 to $F_{i+1}$, we obtain $|D_{i+2}| = 5$ for the component $F_{i+2}$. Denote by $n$ the number of such triples. We see that these triples contain $19n$ vertices of $D$.

Suppose that the component $F_{2k+8}$ belongs to one of the above pairs or triples, and consider a maximal sequence

$$F_{i+1}, F_{i+2}, \ldots, F_{i+r}$$

not containing the components from the above pairs and triples. It is obvious that either $|D_{i+r+1}| = 8$ or $|D_{i+r+1}| = |D_{i+r+2}| = 7$. In the first case we know that $F_{i+r+1}$ is of type D4 and $|D_{i+r}| \leq 6$ by Claim 4. For the latter case we know that $F_{i+r+1}$ must have type B4, C3, D2 or D3. Hence, by Claim 4, $|D_{i+r}| \leq 6$. Thus,

$$\sum_{j=1}^{r} |D_{i+j}| \leq 6.5r.$$

Taking into account all such maximal sequences, we obtain

$$|D \cap V(F)| \leq 13m + 19n + 6.5(2k + 8 - 2m - 3n) = 13k + 52 - 0.5n \leq 13k + 52.$$

Assume now that the component $F_{2k+8}$ does not belong to any of the above pairs or triples, and denote by $L$ a maximal sequence

$$F_{l+1}, F_{l+2}, \ldots, F_{2k+8}$$

not containing the components from those pairs and triples. If $|D_{2k+8}| = 8$, then $|D_{2k+7}| = 6$ by Claim 4. We have

$$\sum_{j=1}^{2k+8-l} |D_{l+j}| \leq 6.5(2k + 8 - l) + 1.5 = 6.5|L| + 1.5.$$

If $|D_{2k+8}| = 7$, then it is not difficult to see that

$$\sum_{j=1}^{2k+8-l} |D_{l+j}| \leq 6.5(2k + 8 - l) + 1 = 6.5|L| + 1.$$

We have already proved that if $F_{i+1}, F_{i+2}, \ldots, F_{i+r}$ ($i + r < 2k + 8$) is a maximal sequence not containing the components of the pairs and triples, then

$$\sum_{j=1}^{r} |D_{i+j}| \leq 6.5r.$$

Taking into account all such maximal sequences and $L$, we obtain

$$|D \cap V(F)| \leq 13m + 19n + 6.5(2k + 8 - 2m - 3n - |L|) + 6.5|L| + 1.5 =$$
Thus, 
\[ |D \cap V(F)| \leq 13k + 53, \]
as required. The proof of Proposition 1 is complete.

**Lemma 5** \( IR(W_k) - \Gamma(W_k) \geq k + 1. \)

**Proof:** Since \( D \) is a \( \Gamma \)-set, it follows that \( D \) is maximal irredundant. Adding to \( D \setminus V(F) \) some new vertices, we will construct a set \( D' \) which is maximal irredundant and 
\[ |D' \cap V(F)| \geq 14k + 54. \]

We first put \( D' = D \setminus V(F) \). Taking into account the definition of the 16 types of the component \( F_1 \), we consider 4 cases. Suppose that \( k' \in D \) and \( k \in PN(k', D) \). Assume that \( k' \notin D \) and \( k' \in N(D \setminus V(F_1)) \), say \( k' \) is adjacent to \( k'' \). Now, we put \( a, b, x, m, n, y \in D' \) if \( \{k'\} = PN(k'', D) \), and we put \( h, i, j, k, m, n, p \in D' \) otherwise. Finally, suppose that \( k' \notin D \) and \( k' \in PN(k, D) \). We put \( h, i, j, k, m, n, p \in D' \).

Let us consider the component \( F_{2k+8} \). Suppose that \( g' \in D \) and \( g \in PN(g', D) \). We put \( a, b, x, m, n, y \in D' \). Assume that \( g' \in D \) but \( g \notin PN(g', D) \). We put \( a, b, x, m, n, y \in D' \). Consider now the case \( g' \notin D \) and \( g' \in N(D \setminus V(F)) \), say \( g' \) is adjacent to \( g'' \). We put \( a, b, x, m, n, y \in D' \) if \( \{g'\} = PN(g'', D) \), and we put \( a, b, c, d, e, f, g \in D' \) otherwise. Finally, suppose that \( g' \notin D \) and \( g' \in PN(g, D) \). We put \( a, b, c, d, e, f, g \in D' \).

For \( 2 \leq i \leq 2k + 7 \), we put \( a, b, c, d, e, f, g \in D' \) if \( i \) is even, and \( h, i, j, k, m, n, p \in D' \) if \( i \) is odd. It is easy to see that the resulting set \( D' \) is a maximal irredundant set and 
\[ |D' \cap V(F)| \geq 14k + 54. \]

Applying Proposition 1, we obtain 
\[ IR(W_k) - \Gamma(W_k) \geq |D'| - |D| = |D' \cap V(F)| - |D \cap V(F)| \geq 14k + 54 - 13k - 53 = k + 1. \]

Using Lemmas 1–5 we can easily choose the integer \( k \) such that the conditions of Theorem 1 are satisfied. The proof of Theorem 1 is complete.

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**References**


