A New Interaction Force Decomposition Maximizing Compensating Forces under Physical Work Constraints

Alexander M. Schmidts, Manuel Schneider, Angelika Peer

Institute of Automatic Control Engineering
D-80290 Munich, Germany
fax: +49-89-289-28340
e-mail: Alexander.Schmidts@tum.de, Angelika.Peer@tum.de
http://www.lsr.ei.tum.de

Abstract: Decomposition of interaction forces in manipulation tasks has a long research tradition. Interaction forces are often split into robustness-reflective and accelerating forces. While this decomposition is typically performed for the synthesis of interaction forces to be applied for example in the context of robotic grasping, less attention has been paid to the analysis of measured, human interaction forces. Here we present a new decomposition approach for interaction force analysis. It extends the intuitive solution known in literature for the two finger grasp and combines it with a physically motivated bounding constraint, which allows the maximization of robustness reflective forces. Advantages of our approach are illustrated with an example and are compared to existing decomposition approaches. In contrast to existing approaches the new approach is not limited in the number of interaction points and incorporates forces which are physically possible only.

1 INTRODUCTION

Grasping, as a frequently used and complex skill, has caught attention in robotics since the 70’s. In general the grasping task involves manipulation of an object by applying task-dependent and multi-purpose interaction forces that accelerate or deform the object. Consequently, interaction forces (IFs) can be decomposed into compensating forces (CFs), also called grasping forces, and manipulating forces (MFs). A CF is the component of an IF which has, combined with the other CFs, no effect on the acceleration of the object. They rather introduce stability and robustness to the grasp. A MF, on the other hand, is the component of an IF which accelerates the object. This composition of IFs is used in robotic grasping for IF generation, also called IF synthesis [1,2]. In contrary, we aim for the decomposition of measured IFs, also called IF analysis, which is of great interest in a series of research areas ranging from joint object manipulation to human grasp analysis [3–7].

In case of IF analysis, the decomposition of given IFs into CFs and MFs requires solving an under-determined system of equations. Thus, a meaningful solution has to be found from the infinite number of possible solutions by making additional assumptions and thus, reducing the solution space. In contrast, IF synthesis requires the composition of IFs from CFs and MFs, which have been derived based on additional requirements, e.g. grasp robustness by Aicardi et al. [8]. For robotic grasping CFs and MFs are often controlled separately. Consequently, IF analysis and synthesis are used simultaneously. The measured IFs are decomposed using IF analysis to be able to calculate an error, while the reference values are determined using IF synthesis.

Yoshikawa and Nagai used intuitive constraints to determine CFs and MFs from given IFs for two, three and four finger grasps [9]. But this decomposition has the disadvantage that it abstracts the interaction to points and, thus, allows no torques to be applied on the object. An approach including this possibility is the virtual linkage model introduced by Williams and Khatib [10]. They solved the under-determined system using the Moore-Penrose Pseudoinverse which leads to the solution with the smallest norm.

Bicchi detailed the composition of IFs and introduced a calculation scheme for the decomposition of forces during whole body manipulation that incorporates body parts like wrist, elbow or hip [11]. He describes the IFs as a sum of
If wrenches resulting from the IFs exist, which compensate each other, the object is squeezed, stretched or distorted and an internal wrench describing the mechanical stress inside the object, also called internal forces [14], evolves. We call components of the IFs with this property compensating forces (CFs) and refer to them with \( f_{c,i} \) throughout this paper. If wrenches resulting from an IF are not compensated, they accelerate the object and an external wrench describing the motion of the object, also called external forces [14], evolves. We call components of the IFs with this property manipulating forces (MFs) and refer to them with \( f_{m,i} \).

The MFs generate a resulting wrench \( \mathbf{w}_r \) acting on the object with

\[
\mathbf{w}_r = (\mathbf{f}_r \times \mathbf{\tau}_r) = \sum_{i=1}^{N} (\mathbf{f}_i \times \mathbf{f}_i) = \sum_{i=1}^{N} \mathbf{w}_i = \mathbf{W} \mathbf{f}
\]

\[
\mathbf{w}_m = \sum_{i=1}^{N} (\mathbf{f}_{m,i} \times \mathbf{f}_{m,i}) = \sum_{i=1}^{N} \mathbf{w}_{m,i} = \mathbf{W} \mathbf{f}_m
\]

where \( \mathbf{f}_r \) is the accelerating force, \( \mathbf{\tau}_r \) the accelerating torque and

\[
\mathbf{W} = \begin{bmatrix} \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I} \\ \mathbf{R}_1 & \mathbf{R}_2 & \cdots & \mathbf{R}_N \end{bmatrix},
\]

\[
\mathbf{f}_m = [\mathbf{f}_{m,1}^T, \mathbf{f}_{m,2}^T, \cdots, \mathbf{f}_{m,N}^T]^T,
\]

\[
\mathbf{f} = [\mathbf{f}_1^T, \mathbf{f}_2^T, \cdots, \mathbf{f}_N^T]^T,
\]

\[
\mathbf{w}_i = \left( \mathbf{f}_i \times \mathbf{f}_i \right) \quad \text{and} \quad \mathbf{w}_{m,i} = \left( \mathbf{r}_i \times \mathbf{f}_{m,i} \right),
\]

whereby \( \mathbf{I} \) is the identity matrix and \( \mathbf{R}_i \) the skew symmetric matrix operator of \( \mathbf{r}_i \) performing the cross product.

The CFs, on the other hand, generate internal wrenches \( \mathbf{w}_{c,i} \) and following their definition they sum up to zero. It follows

\[
\sum_{i=1}^{N} (\mathbf{f}_{c,i} \times \mathbf{f}_{c,i}) = \sum_{i=1}^{N} \mathbf{w}_{c,i} = \mathbf{W} \mathbf{f}_c = \mathbf{0}
\]

with

\[
\mathbf{f}_c = [\mathbf{f}_{c,1}^T, \mathbf{f}_{c,2}^T, \cdots, \mathbf{f}_{c,N}^T]^T \quad \text{and} \quad \mathbf{w}_{c,i} = \left( \mathbf{f}_{c,i} \times \mathbf{r}_i \right).
\]

Using (1), (2) and (3) an under-determined system of equations is defined.

### 2.2 Related Work

In literature different solutions to this under-determined system have been proposed and in the following paragraphs we...
Figure 2: Two finger grasp example without gravity. (cp. [9])

will have a more detailed look at two representative solutions, the virtual linkage model of Williams and Khatib [10] and the more intuitively derived approach of Yoshikawa and Nagai [15].

For a better comprehension of the virtual linkage model the example shown in Fig. 2 will be used with

\[ f_{m,1} = f_1; \quad f_{c,1} = 0; \quad \text{and} \quad f_{m,2} = 0, \quad f_{c,2} = 0. \]

The general solution of the virtual linkage model for the resulting MFs is

\[ f_m = W^T (W W^T)^{-1} W f \]

which reduces to

\[ f_{m,i} = \frac{1}{N} f_r + R_i \left( \sum_{j=1}^{N} R_j \right)^{-1} \tau_r \quad \text{if} \quad \sum_{i=1}^{N} r_i = 0. \]  

It can be shown that the solution to (4) is invariant to shifts of the object-fixed coordinate system and thus, after shifting the reference frame to \( \sum_i r_i = 0 \), (5) can be considered a simplified solution of (4). From (5) it can be seen that the resulting force \( f_r \) on the object is distributed equally on all MFs. That means even if an IF has no influence on the acceleration of the object it is assigned a MF larger than zero. We call these forces virtual forces because they are physically impossible and therefore non-existent. For the example in Fig. 2 this means \( f_{m,2} = f_1/2 \neq 0 \).

The approach of Yoshikawa and Nagai in contrary does not lead to virtual forces, but has other drawbacks. Their method is based on three intuitive assumptions: First, CFs should always be inside the friction cone. Second, a MF should have no part pointing into the inverse direction of the corresponding CF. Third, a MF has no part resulting in compression or tension of the object, neglecting torsion. From these assumptions follow two steps for IF decomposition.

In the first step, possible grasp modes \( \alpha = [\alpha_1, \cdots, \alpha_m] \) with \( \alpha_i \in \{-1; 1\} \) have to be chosen. A grasp mode describes if CFs between two interaction points squeeze or stretch the object and depend on the surface normals and the friction coefficients at the interaction points. A grasp mode can be calculated for the three finger grasp by using the algorithm described in [9]. The CFs are described in a subspace \( h_m \) using these grasp modes. In this subspace a solution is only feasible if all values are positive. Otherwise, the grasp mode would define compression, while the subspace value would result in tension.

In the second step, given a grasp mode, different solutions to the MFs, again described in an own subspace \( h_m \), are tested for feasibility, i.e. no MF results into tension or compression of the object neglecting torsion and no MF points into the inverse direction of its corresponding CF. The different solutions result from any perturbation of a selection vector \( k = [k_1, \cdots, k_l] \) with \( k_i \in \{0; 1\} \) that selects possible directions for the MFs.

From the above considerations the following system of equations results:

\[ w_r = W f = WBh \quad \Rightarrow \quad h = B^{-1} f \]

with \( B = [B_c(\alpha, r_1, \cdots, r_n) \quad B_m(k, \alpha, r_1, \cdots, r_n)] \),

\[ h = \begin{bmatrix} h_c \\ h_m \end{bmatrix} \]

After calculating all solutions for the MFs from the perturbations of \( k \), the feasible solution, if one exists, has to be found by testing if the subspace values fulfill the assumptions. Using the resulting \( h \) the CFs and MFs can be calculated by

\[ f_c = B_c h_c, \quad f_m = B_m h_m. \]

This approach has multiple drawbacks. For example it is possible that multiple grasp modes (see [9] for examples) and eventually multiple solutions exist or that no grasp mode exists. There may be also no selection vector which leads to a feasible solution and thus, decomposition. This is due to the constraint requiring that a MF is composed of forces which do not lead to tension or compression for selected parts of the corresponding CF (cp. Condition 3 for MFs in [9]) which also reduces the solution space to the empty set for most grasp configurations with four fingers. Furthermore, only algorithms are given to determine the grasp mode for two and three finger grasps, because the approach gets very complex when additional interaction points are added. It remains also unclear if a solution for more than four fingers exists.

### 2.3 Proposed IF Decomposition

So far the mathematical decomposition into MFs and CFs is based on the definitions given in Section 2.1. We will extend these definitions to allow physically possible forces only. For this reason we introduce the following bounding constraint.

**Bounding constraint:** This constraint is inherently included in the verbal definition of the MFs, but has not been motivated in literature yet by the best knowledge of the authors. Because a MF is the part of an IF, which accelerates the object it also does physical work resulting in a differential change in energy of the object. Thus, by the law of conservation of energy the differential change in energy resulting from the MF cannot be larger than the one resulting
Figure 3: The visualized solution space given by (6). Every vector combination consists of $f_{m,i}$ and $f_{c,i}$.

from its corresponding IF. This is stated in Lemma 1 (see Sec. 5 for proof).

**Lemma 1.** Considering that a MF is the part of its IF, which performs physical work and taking (1) into account it follows

$$f_{m,i}^T f_{m,i} + f_{c,i}^T f_{c,i} \leq f_1^T f_1.$$  \hspace{1cm} (6)

**Remark:** The inequality constraint (6) bounds the solution space for the CFs and the MFs to a sphere around their respective IFs with radius $|f_1|/2$ as illustrated in Fig. 3 for two dimensions.

Given the bounding constraint the solution space is reduced to physically possible forces, but still an infinite number of solutions exist. A first intuitive approach to solve this problem would be to define that MFs contribute only to the acceleration of an object without any compensating parts, which would lead to a full decomposition. However, torques applied on the object at the single interaction points can only be orthogonal to the corresponding vector $r_i$ (cp. (2)). Furthermore, considering that the resulting torque mostly points into a direction, which is not orthogonal to any of the $r_i$, it is likely that for the specific studied situation no solutions for the full decomposition problem exists. From this follows Lemma 2 (see Sec. 5 for proof).

**Lemma 2.** Full decomposition of IFs into CFs leading to wrenches compensating each other and MFs contributing to the resulting wrench only, i.e. without compensating parts, is in general not possible.

Therefore, we adopt an intuitive approach for the two finger grasp originally formulated by Yoshikawa and Nagai [9], and extend it to more than two fingers. Yoshikawa and Nagai propose to calculate the internal forces for a two finger grasp based on

$$f_{c,1/2} = \pm \min (|f_1^T e_{12}|, | - f_2^T e_{12}|) e_{12} \quad \text{with} \quad (7)$$

$$e_{12} = \frac{r_2 - r_1}{|r_2 - r_1|},$$  \hspace{1cm} (8)

where $e_{12}$ represents the unit vector from one interaction point to the other one. It should be noted that (7) holds also for objects with holes. In (7) the interaction forces are projected on the line connecting the interaction points and the smaller projected force is chosen as compensating component. This is due to the fact that both CFs have to compensate each other, which means that their norms have to be equal and thus, only the smaller norm can be fully compensated. In other words, the CFs are maximized which is the first property we abstract from this approach. Extending this idea to multiple fingers we propose to design a cost function that maximizes the CFs. Mathematically, the cost function can be established in multiple ways, e.g. by minimizing the MFs, which are not contributing to the resulting wrench or by maximizing the CFs. Following up on this idea, the solution space of the CFs has to be bounded so that they do not increase to infinity. In the approach by Yoshikawa and Nagai this is achieved by the proposition that a MF should have no part pointing into the inverse direction of the corresponding CF and vice versa, i.e. $f_{c,i}^T f_{m,i} \geq 0$. Utilizing (1) we can derive the previously motivated bounding constraint (6) and get a second property. A last property is given by the fact that CFs can only be applied along the line connecting the two interaction points, which is the only solution to (3) for the two finger grasp [16]. Hence, using (3) additionally to the bounding constraint allows us to expand the approach of Yoshikawa and Nagai to more than two fingers.

**IF Decomposition Theorem (IFDT).** For a precision grasp the IF decomposition problem is given by the following optimization problem:

$$\arg \max_{f_{c,i}} \quad J = |f_c|^2$$

s.t. $$W f_c = 0,$$  \hspace{1cm} (10)

$$f_{c,i}^T f_{c,i} \leq f_1^T f_1 \quad \forall i.$$  \hspace{1cm} (11)

Please note that inequality (11) is obtained by inserting (1) into (6).

**Remark:** It can be shown that the solution for the two-finger grasp equals the solution proposed by Yoshikawa and Nagai as shown in Lemma 3 (see Sec. 5).

### 3 Numerical Examples

For illustration and comparison of the new decomposition approach with state-of-the-art approaches, namely the approach of Yoshikawa and Nagai [9] and the virtual linkage model [10], the example of Fig. 4 is adopted with

$$f_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T,$$  \hspace{1cm} $r_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}^T,$

$$f_2 = \begin{bmatrix} -\varepsilon \\ -1 \end{bmatrix}^T,$$  \hspace{1cm} $r_2 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}^T,$

$$f_3 = \begin{bmatrix} \varepsilon \\ -1 \end{bmatrix}^T,$$  \hspace{1cm} $r_3 = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}^T,$

and some $\varepsilon \in \mathbb{R}$. Values in $z$-direction are assumed to be zero and it should be noted that $r_1 + r_2 + r_3 = 0$ so that
A New Interaction Force Decomposition Maximizing Compensating Forces under Physical Work Constraints

Figure 4: A three finger grasp.

(5) can be used directly. The decomposition based on the newly proposed IFDT was performed using the optimization toolbox of MATLAB adopting an interior-point algorithm, which is suitable for quadratic optimization problems with nonlinear equality and inequality constraints.

Assuming $\varepsilon = 0$, $f_2$ and $f_3$ are compensated by $f_1$ and the only force influencing the objects motion is the $x$ component of $f_1$. The MFs and CFs of the virtual linkage model (vl) can be determined using (5) and (1) and are given by:

$$vl\ f_{m,1} \approx \begin{bmatrix} 0.67 \\ 0 \end{bmatrix}^T,$$
$$vl\ f_{c,1} \approx \begin{bmatrix} 0.33 \\ 2 \end{bmatrix}^T,$$
$$vl\ f_{m,2} \approx \begin{bmatrix} 0.17 \\ 0.29 \end{bmatrix}^T,$$
$$vl\ f_{c,2} \approx \begin{bmatrix} -0.17 \\ -1.29 \end{bmatrix}^T,$$
$$vl\ f_{m,3} \approx \begin{bmatrix} 0.17 \\ -0.29 \end{bmatrix}^T,$$
$$vl\ f_{c,3} \approx \begin{bmatrix} -0.17 \\ -0.71 \end{bmatrix}^T.$$

From this follows that

$$vl\ f_{m,1}^Tvl\ f_{m,2} + vl\ f_{c,2}^Tvl\ f_{c,2} > vl\ f_{1}^Tvl\ f_{2},$$

which is contradicting (6) and, thus, for $f_{m,2,vl}$ virtual forces are calculated. Also $f_{m,3,vl}$ would contain virtual forces if the influence of $f_1$ on the object’s motion gets larger. Summarizing, for the virtual linkage model we can conclude that if the object is accelerated mostly by a specific IF and the accelerating forces (MFs) are much larger than the stabilizing forces (CFs), the decomposed MFs will primarily be virtual.

Using the approach of Yoshikawa and Nagai the grasp modes shown in Fig. 5 result. However, if $\varepsilon \in [0.356; 0.577]$ there exists no solution because the constraint prohibiting the MFs to lead to tension or compression along a joining line cannot be fulfilled. Thus, depending on the possible grasp modes there may always be regions where no decomposition is possible. In contrast to these approaches the IFDT leads to the following MFs and CFs, which are respecting the bounding constraint:

$$IFDT\ f_{m,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T,$$
$$IFDT\ f_{c,1} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}^T,$$
$$IFDT\ f_{m,2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T,$$
$$IFDT\ f_{c,2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}^T,$$
$$IFDT\ f_{m,3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T,$$
$$IFDT\ f_{c,3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}^T.$$

Figure 5: Possible grasp modes for the example shown in Fig. 4 using the friction constant $\mu = 0.9$. $\alpha = [- + +]$ means that the CF between the interaction points $r_2$ and $r_3$ is stretching. $\alpha = [+ + +]$ means that all CF are squeezing.

4 CONCLUSION

We have introduced a new approach for the decomposition of IFs into MFs and CFs for IF analysis. For this purpose an intuitive approach originally introduced by Yoshikawa and Nagai for the two finger grasp has been formalized and extended to more than two interaction points resulting in an optimization problem, which maximizes CFs. This maximization is only possible due to a new introduced bounding constraint, which bounds the solution space of the MFs and the CFs. The constraint is motivated by considering that a force component cannot do more physical work than the original interaction force.

Existing decomposition approaches, the virtual linkage model and the approach of Yoshikawa and Nagai, were compared to the newly proposed IF decomposition. While the virtual linkage model was found to lead to impossible MFs, when taking into account the law of conservation of energy for each interaction force separately, the main drawback of the method of Yoshikawa and Nagai was found that it is not clear if there exists a solution for more than four interaction points.

Since the optimization problem contains a quadratic constraint which complicates an analytical solution, our future work will target a numerical solution for online decomposition of IFs.

5 APPENDIX

Lemma 1. Considering that a MF is the part of its IF which performs physical work and taking (1) into account it follows

$$f_{m,i}^T f_{m,i} + f_{c,i}^T f_{c,i} \leq f_i^T f_i.$$  

(12)
Adding (17) and (18) results in inequality (12).

\[
\begin{aligned}
\mathbf{f}_{\text{i,proj}} &= \left( f^T_{\text{i,proj}} \mathbf{f}_{\text{m,i}} \right) \mathbf{f}_{\text{m,i}} \\
0 &\leq f^T \mathbf{r}_{\text{i}} d\mathbf{s} \leq f^T_{\text{i,proj}} \mathbf{r}_{\text{i}} d\mathbf{s}
\end{aligned}
\]

(13)

can be used to describe an upper bound for the physical work performed by the MF. Second, if the work performed by the IF is positive/negative also the work done by the corresponding MF must be positive/negative. From these statements the following two inequalities can be formulated which must hold for every infinitesimal line segment \(d\mathbf{r} = d\mathbf{s} = 1\)

\[
0 \leq f^T \mathbf{r}_{\text{i}} d\mathbf{s} \leq f^T_{\text{i,proj}} \mathbf{r}_{\text{i}} d\mathbf{s}
\]

(14)
or rotary segment \(d\phi = q d\varphi\) with \(|q| = 1\)

\[
0 \leq [\mathbf{r}_{\text{i}} \times \mathbf{f}_{\text{m,i}}]^T q d\varphi \leq [\mathbf{r}_{\text{i}} \times f^T_{\text{i,proj}}] q d\varphi.
\]

(15)

Substituting (13) into (14) and (15) yields

\[
0 \leq f^T \mathbf{r}_{\text{i}} d\mathbf{s} \leq c_i f^T \mathbf{r}_{\text{i}} d\mathbf{s},
\]

\[
0 \leq [\mathbf{r}_{\text{i}} \times \mathbf{f}_{\text{m,i}}]^T q d\varphi \leq c_i [\mathbf{r}_{\text{i}} \times \mathbf{f}_{\text{m,i}}]^T q d\varphi
\]

with

\[
c_i = \frac{f^T \mathbf{f}_{\text{m,i}}}{|\mathbf{f}_{\text{m,i}}|^2}.
\]

Comparing the coefficients it follows

\[
0 \leq 1 \leq c_i.
\]

(16)

Substituting \(c_i\) into (16) leads to

\[
0 \leq f^T \mathbf{r}_{\text{i}} d\mathbf{s} \leq |f_{\text{m,i}}|^2 \leq f^T f_{\text{m,i}}.
\]

(17)

Since \(f^T f_{\text{m,i}} \geq 0\) the angle \(\alpha\) between \(f_{\text{r}}\) and \(f_{\text{m,i}}\) is within \([-\pi/2; \pi/2]\). When substituting (1) into (17) follows

\[
f^T f_{\text{r}} f_{\text{c,i}} \geq |f_{\text{c,i}}|^2 = f^T f_{\text{c,i}} f_{\text{c,i}}
\]

(18)

Adding (17) and (18) results in inequality (12).

\[\square\]

**Lemma 2.** Full decomposition of IFs into CFs leading to wrenches compensating each other and MFs contributing to the resulting wrench only, i.e. without compensating parts, is in general not possible.

**Proof.** If a force is not pointing into the direction of \(f_{\text{r}}\) nor its resulting torque into the direction of \(\tau_{\text{r}}\), then this force must have components, which are compensated by other forces. Consequently, full decomposition can only be achieved if the MFs consist of two parts only: one part pointing into the direction of \(f_{\text{r}}\) and a second part denoted as \(f_{\text{c,j}}\), leading to a torque pointing into the direction of \(\tau_{\text{r}}\). From this follows:

\[
\begin{aligned}
f_{\text{m,i}} &= d_1 f_{\text{r}} + f_{\text{x,i}}, \\
r_{\text{i}} \times f_{\text{x,i}} &= d_2 \tau_{\text{r}}
\end{aligned}
\]

(19)

(20)

In (20) the torque on the left side is orthogonal to \(r_{\text{i}}\), while the torque on the right side is pointing into the direction of \(\tau_{\text{r}}\). Thus, a solution to (20) exists only if \(\tau_{\text{r}}\) is orthogonal to \(r_{\text{i}}\). As it is, however, easy to find an example with a force and grasping point constellation comprising a torque \(\tau_{\text{r}}\) that is not orthogonal to \(r_{\text{i}}\), it can be concluded that in general the MFs cannot have only parts contributing to the resulting wrench, but will also contain compensating forces and torques.

\[\square\]

**Lemma 3.** The solution to the two-finger grasp constraint (10) can be written as follows

\[
I R_1 R_2 \begin{bmatrix} f_{\text{c,1}} \\ f_{\text{c,2}} \end{bmatrix} = 0.
\]

(21)

From the first row follows that the two CFs have equal norms and point in opposite directions \(f_{\text{c,2}} = -f_{\text{c,1}}\). Inserting this result into the second row of (21) and rewriting the cross product in its original form gives

\[
(r_1 - r_2) \times f_{\text{c,1}} = 0.
\]

with the trivial solution \(f_{\text{c,1}} = f_{\text{c,2}} = 0\) or \(f_{\text{c,1}}\) and \(f_{\text{c,2}}\) parallel to \(r_1 - r_2\). The non-trivial solution means that both CFs lie on the line connecting the two interaction points, which can be parametrized using \(e_{12}\) from (8):

\[
f_{\text{c,1}} = \alpha e_{12}, \quad f_{\text{c,2}} = -\alpha e_{12} \quad \alpha \in \mathbb{R}
\]

(22)

Thus, the value \(\alpha\) remains to be determined. Using (22) the optimization problem (9)-(11) can be reformulated to

\[
\text{arg max } \alpha \quad J = 2\alpha^2
\]

s.t. \[\alpha^2 \leq \alpha f^T e_{12}, \]

\[\alpha^2 \leq -\alpha f^T e_{12}.
\]

(23)

(24)

(25)

Assume that \(f^T e_{12} > 0\) and \(f^T e_{12} < 0\) and note that the left sides of (24) and (25) are always positive or zero. Then, if \(\alpha \in \mathbb{R}_0^+\), (24) and (25) allow the solution \(\alpha = 0\) only. On the other hand, if \(\alpha \in \mathbb{R}_0^-\) the cost function can take larger values and maximizes the cost function. Similar considerations can be made for all other combinations of signs of
Table 1: Possible sets of $\alpha$ for given projections of the IFs on the line connecting the interaction points.

<table>
<thead>
<tr>
<th>$f_1^T e_{12}$</th>
<th>$f_2^T e_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$\alpha &lt; 0$</td>
</tr>
<tr>
<td>$\alpha &gt; 0$</td>
<td>$\alpha = 0$</td>
</tr>
</tbody>
</table>

$f_1^T e_{12}$ and $f_2^T e_{12}$ summarized in Table 1. Hence, dependent on the IFs and the line connecting the interaction points one out of four possible solutions exist for maximization of (23) under the constraints (24) and (25). Two of them require $\alpha = 0$. The other two can be determined by reformulating (24) and (25) as follows

$$
\begin{align}
\text{if } \alpha > 0 & \Rightarrow \left\{ \begin{array}{l}
\alpha \leq f_1^T e_{12} = k_1 \\
\alpha \leq -f_2^T e_{12} = k_2
\end{array} \right. \\
\text{if } \alpha < 0 & \Rightarrow \left\{ \begin{array}{l}
\alpha \geq f_1^T e_{12} = -k_1 \\
\alpha \geq -f_2^T e_{12} = -k_2
\end{array} \right.
\end{align}
$$

with $k_1, k_2 \in \mathbb{R}_0^+$. If $f_1^T e_{12} > 0$ and $f_2^T e_{12} < 0$, $\alpha$ is positive and must be maximized under constraint (26). Thus, $\alpha$ equals either $k_1$ or $k_2$. Second, if $f_1^T e_{12} < 0$ and $f_2^T e_{12} > 0$, $\alpha$ is negative and must be minimized under constraint (27). Thus, $\alpha$ equals either $-k_1$ or $-k_2$. Combining these results we can state that

$$
\alpha = \pm \min(|f_1^T e_{12}|, |f_2^T e_{12}|).
$$

Thus, by inserting (28) into (22) we get (7).

References


