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Robust and stepwise optimization design for CO₂ pipeline transportation

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Abstract: Carbon capture, utilization, storage (CCUS) technology is an effective means to reduce the CO₂ emissions. It has been noted that engineering and economic design of the pipeline transportation are important components for CCUS. However, the uncertainties of the pipeline transportation model may make an infeasible design in practice and cause unnecessary cost. In this paper, a novel robust optimization method is proposed for CO₂ pipeline transportation design, which can deal with the multiple uncertainties. A stepwise method is presented to further improve the optimization performance. The proposed optimal algorithm is validated by using numerical studies, which show the proposed approach can deal with the multiple uncertainties and improve the design performance in comparison with the existing methods.

Keywords: CO₂ emissions; pipeline transportation; uncertainties; robust optimization; stepwise optimization

Nomenclature

Cₚ_cap capital cost of pipelines
C_c_cap capital cost of compressors
C_b_cap capital cost of booster pumps
C_T_OM annual operation and maintenance costs of pipelines, compressors and booster pumps
C_T_energy annual energy costs of compressors and booster pumps
CRFᵢ capital recovery factor of pipelines
24 $CRF_2$ capital recovery factor of compressors
25 $CRF_3$ capital recovery factor of booster pumps
26 $C_{ps}$ price of steel pipeline
27 $C_{fg}$ price of electricity
28 $D_{inom}$ inner diameter of the pipeline
29 $E$ longitudinal joint factor
30 $F$ design factor.

31 $f_\rho(P_{ave}, T_{ave})$ function of density that depends on the $P_{ave}$ and $T_{ave}$
32 $f_\mu(P_{ave}, T_{ave})$ function of viscosity that depends on the $P_{ave}$ and $T_{ave}$
33 $f_M$ material cost factor
34 $f_{BO\%M}$ percentage of the capital cost of booster pumps
35 $f_{PO\%M}$ percentage of pipelines capital cost
36 $f$ Darcy-Weisbach friction
37 $H_{op}$ operation time of the transportation
38 $I_0$ base costs for calculating the compressor capital cost
39 $ID_{id}$ optimal ideal inner diameter
40 $ID_{NPS}$ inner diameter of the NPS
41 $L$ length of the pipeline
42 $L_{pum}$ maximum distance between the boosting pump stations
43 $LC$ levelized cost of CO$_2$ pipeline transportation
44 $M$ molar mass
45 $N$ total number of compression stages
46 $N_{pump}$ number of boosting pump stations
47 $N_{Boost}$ largest number of boosting pump stations
48  \( n \)      multiplication exponent
49  \( O & M \)    operation and maintenance costs
50  \( P_{\text{inlet}} \)  inlet pressure for each pipe segment
51  \( P_{\text{Outlet}} \)  outlet pressure for each pipe segment
52  \( P_{\text{max}} \)  maximum pressure for CO\(_2\) transportation
53  \( P_{\text{inj}} \)    injection pressure
54  \( P_{\text{ave}} \)  average pressure along the pipeline
55  \( P_{\text{mop}} \)  maximum allowable operation pressure
56  \( P_{\text{cap}} \)  suction pressure
57  \( P_{\text{MOP}} \)  discharge pressure
58  \( Q_{\text{m}} \)    CO\(_2\) mass flow rate
59  \( R \)      universal gas constant
60  \( r \)      discount rate
61  \( S \)      specified minimum yield stress for the pipe material
62  \( T_{\text{ave}} \)  average temperature
63  \( T_{\text{soil}} \)  soil temperature around the pipeline
64  \( T_{\text{mop}} \)  largest soil temperature around the pipeline
65  \( T_{C} \)    operation time of compressor
66  \( T_{B} \)    operation time of booster pump
67  \( T_{i} \)    suction temperature
68  \( t \)      wall thickness
69  \( t_{\text{id}} \)  optimal ideal wall thickness
70  \( t_{\text{NPS}} \)  wall thickness of the NPS
71  \( V \)      actual velocity
1 Introduction

The process of CO2 capture, transportation, enhanced oil recovery (EOR) and storage is one of the best ways to reduce the CO2 emissions, which not only can effectively prevent the increase of CO2 concentration in the atmosphere, but also bring economic benefit with EOR (Marston 2013; Imtiaz et al. 2015). As a link between the capture source and the storage site, pipelines are attractive approach to transport large amount of CO2 for long distance (Svensson et al. 2004; Luo et al. 2014; Martynov et al. 2015). It is obvious that engineering and economic pipeline design are important for the CCUS technology (Knoope et al. 2013). Most of the existing optimal approaches are model based, whose performance is affected by the uncertainties seriously (Zhang et al.
Ignoring the uncertainties may lead to an infeasible design. Consequently, it is central to consider the uncertainties in optimization design for the pipeline transportation.

There are mainly two types of uncertainty of the CO₂ pipeline transportation: (1) Engineering model uncertainties. The CO₂ temperature along the pipeline changes with the seasons, even the pipeline has insulation and is buried in the soil (Zhang et al. 2012). Along with time, booster pumps, compressors and other equipment will be aging and their performance parameters will be varying. The impurities in CO₂ such as H₂S, SOX and O₂ impact the density and viscosity. (2) Economic model uncertainties. There are many changes in labor, material, land prices, and regulations in pipeline lifetime (Middleton 2013). For easy calculation, the electricity cost is usually assumed to be constant over time. However, it is an significant uncertainty in the costs of CCUS power plants and in the electricity cost (Knoope et al. 2014). If these uncertainties are not considered in the pipeline transportation design, it may degrade the design performance.

To design the pipeline transportation easily, the existing researches always assume the temperature is constant along the pipeline (Chandel et al. 2010; Gao et al. 2011; Knoope et al. 2014), however, the temperature variation can significantly affect the transport cost (Chandel et al. 2010) and/or make the design not well (Zhang et al. 2012). To solve this problem, the highest temperature of the soil is used in the pipeline design. In order to simplify the calculation of the pipeline design, the linear optimization is introduced (Morbee et al. 2012; Middleton 2013), in which the modelling uncertainty is not considered. Effects of geologic reservoir uncertainties are analyzed on CO₂ transportation (Middleton et al. 2012), Monte Carlo trials are used to assess the sensitivity of transport cost to the uncertain model parameters (McCoy et al. 2008), but these approaches (McCoy et al. 2008; Middleton et al. 2012) have not discussed the design issues. Considering some uncertainties, an iteration method is proposed for the pipeline design (Knoope et al. 2015). However, the method is based on the designer’s experience, which unavoidably exists one-sidedness in the design. In summary, the effects of uncertainties should not be ignored, the
existing methods focus on the effects of single uncertainty (Chandel et al. 2010) or partial uncertainties (McCoy et al. 2008; Middleton et al. 2012; Knoope et al. 2015). But these approaches usually lack theoretical analysis. There have not been effective methods to deal with the multiple uncertainties. Therefore, it is necessary to present an approach to cope with multiple uncertainties and improve the design performance. The final selected inner diameter and wall thickness are the nominal pipe size (NPS) in the engineering practice which are larger than the ideal ones in general (McCoy et al. 2008; Zhang et al. 2012), therefore, in order to further improve the design performance, a new algorithm is desired to be explored.

In this paper, multiple uncertainties are transformed into bounded set, a new robust optimization model is initially developed to minimize the levelized cost of the CO$_2$ pipeline design, which is solved by using the linear matrix inequalities (LMI). The proposed robust optimization approach can deal with the effects of multiple uncertainties, which not only include the variable temperature, declined parameter performance, changeable density and viscosity, but also the change in labor, material, land prices etc. A stepwise optimization following the robust optimization is provided to further improve the optimization performance. The proposed approach is validated by using numerical studies. It should be mentioned that this paper further improves the results of (Zhao et al. 2016), which has not considered the effects of multiple uncertainties.

The rest of this paper is organized as follows. Uncertain optimization problem is formulated in Section 2. Solutions for robust optimization issue are given in Section 3. The stepwise optimization is presented in Section 4. The computation results and analysis are presented in Section 5. Finally, the conclusions are drawn in Section 6.

2 Uncertain optimization problem description

The optimal design for CO$_2$ pipeline transportation includes the inlet pressure, inner diameter, wall thickness and the number of boosting pump stations. Levelized cost and inlet pressure are selected as the objective function and design variable in the pipeline design, respectively.
The optimization model of CO\textsubscript{2} transportation is formulated as follows (Knoope et al. 2014):

\[
\begin{align*}
\text{min} & \quad LC \\
\text{s.t.} & \quad P_{\text{outlet}} < P_{\text{inlet}} < P_{\text{max}} \\
& \quad V_{\text{min}} < V < V_{\text{max}} \\
& \quad P_{\text{act}} = P_{\text{inlet}} - \Delta P_{\text{act}} L / (N_{\text{pump}} + 1) \\
\end{align*}
\]

\[LC = CRF_{1} \times C_{P_{\text{cap}}} + CRF_{2} \times C_{C_{\text{cap}}} + CRF_{3} \times C_{B_{\text{cap}}} + C_{T_{\text{OM}}} + C_{T_{\text{energy}}} \times Q_{m} \times 10^{-3} \times H_{\text{ope}} \times 3600\]

\[CRF_{x} = \frac{r}{1-(1+r)^{-z_{x}}}\]

where \( LC \) is the levelized cost of CO\textsubscript{2} pipeline transportation (€ / t CO\textsubscript{2}); \( P_{\text{inlet}} \) is the inlet pressure for each pipe segment, which is selected as a decision variable (MPa); \( P_{\text{outlet}} \) is the outlet pressure for each pipe segment (MPa); \( P_{\text{max}} \) is the maximum pressure for CO\textsubscript{2} transportation (MPa); \( V, V_{\text{min}}, V_{\text{max}} \) are the actual, minimum and maximum velocities, respectively (m / s); \( \Delta P_{\text{act}} \) is the actual pressure drop (MPa/m); \( L \) is the length of the pipeline (m); \( N_{\text{pump}} \) is the number of boosting pump stations; \( C_{P_{\text{cap}}}, C_{C_{\text{cap}}}, C_{B_{\text{cap}}} \) are the capital costs of pipelines, compressors and booster pumps, respectively (€); \( C_{T_{\text{OM}}} \) are the annual operation and maintenance (O&M) costs of pipelines, compressors and booster pumps (€); \( C_{T_{\text{energy}}} \) are the annual energy costs of compressors and booster pumps (€). \( Q_{m} \) is the CO\textsubscript{2} mass flow rate (kg/s); \( H_{\text{ope}} \) is the operation time of the transportation (hour/year); \( CRF_{1}, CRF_{2}, CRF_{3} \) are the capital recovery factors of pipelines, compressors and booster pumps, respectively (x = 1, 2, 3); \( r \) is the discount rate (%); \( z_{1}, z_{2}, z_{3} \) are the lifetime of pipelines, compressors and booster pumps, respectively (years). In order to make the paper clearly and easily to follow, Table 1 shows the detail models and the related literatures of \( C_{P_{\text{cap}}}, C_{C_{\text{cap}}}, C_{B_{\text{cap}}}, C_{T_{\text{OM}}}, C_{T_{\text{energy}}} \).

Table 1 Detail models and the related literatures

<table>
<thead>
<tr>
<th>Model</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{P_{\text{cap}}} )</td>
<td>(Gao et al. 2011; Knoope et al. 2013)</td>
</tr>
<tr>
<td>( C_{C_{\text{cap}}} )</td>
<td>(Knoope et al. 2014)</td>
</tr>
</tbody>
</table>
Based on the first order Taylor series, the objective function can be linearized as:

\[ LC = AP_{\text{inlet}} + b \]  

(4)

where \( A \) is a coefficient, \( b \) is a constant.

Considering the uncertainties, (4) can be written as:

\[ LC = (A_0 + \delta_{A_0})P_{\text{inlet}} + (b_0 + \delta_{b_0}) \]  

(5)

where \( A_0 \) and \( b_0 \) are nominal, \( \delta_{A_0} \) and \( \delta_{b_0} \) are uncertainties, respectively.

Squaring equation (5), the optimization model can be re-written as:

\[
\begin{align*}
\min \quad & P_{\text{inlet}}^2 & & \sim \max_{u_{\infty}} \sim \max_{u_{\infty}} \sim \max_{u_{\infty}} \\
\text{s.t.} \quad & P_{\text{inlet}} < P_{\text{outlet}} < P_{\text{max}} & & V_{\min} < V < V_{\max} \\
& P_{\text{outlet}} = P_{\text{inlet}} - \Delta P_{\text{inlet}} L/(N_{\text{pump}} + 1)
\end{align*}
\]

where \( \sum_{i=1}^{l} u_i A_i = \delta_{A_0}, \sum_{i=1}^{l} u_i B_i = \delta_{b_0} + A_0 \delta_{A_0} + \delta_{A_0} \delta_{b_0} = \delta_{b_0}, \sum_{i=1}^{l} u_i C_i = 2b_0 \delta_{b_0} + \delta_{b_0}^2 = \delta_{c_0} \). \( A_0, B_0, C_0 \) are nominal parameters, \( B_0 = A_0 b_0 \), \( C_0 = b_0^2 \cdot \sum_{i=1}^{l} u_i A_i = \delta_{A_0}, \sum_{i=1}^{l} u_i B_i = \delta_{b_0} + A_0 \delta_{A_0} + \delta_{A_0} \delta_{b_0} = \delta_{b_0}, \sum_{i=1}^{l} u_i C_i = 2b_0 \delta_{b_0} + \delta_{b_0}^2 = \delta_{c_0} \). \( A_i, B_i, C_i \) are the uncertainty directions; \( u = [u_1, u_2, \cdots] \) are the uncertainties, \( \|u\|_\infty \leq \varepsilon \).

In order to realize the minimization of \( LC \) under the uncertainties, (6) can be rewritten as:

\[
\begin{align*}
\min \quad & \max_{P_{\text{inlet}}} P_{\text{inlet}}^2 & & \sim \max_{u_{\infty}} \sim \max_{u_{\infty}} \sim \max_{u_{\infty}} \\
\text{s.t.} \quad & P_{\text{inlet}} < P_{\text{outlet}} < P_{\text{max}} & & V_{\min} < V < V_{\max} \\
& P_{\text{outlet}} = P_{\text{inlet}} - \Delta P_{\text{inlet}} L/(N_{\text{pump}} + 1)
\end{align*}
\]

(7)

where \( \|u\|_\infty = \max|u_i| \). (7) is named as robust optimal model of the original issue. The detailed formations of the robust models are given in Section 3 for the stepwise optimization.

**Remark 1:** \( \|u\|_\infty \leq \varepsilon \) denotes the uncertainties which is bounded.
Additional engineering models

The diameter is an intermediate variable, it can be calculated with the flow rate, variable pressure and temperature along the pipeline, it is implied in $C_{p_{cap}}$ and $C_{T_{OM}}$ of the objective function (Gao et al. 2011; Knoope et al. 2013; Knoope et al. 2014):

Pipeline inner diameter can be calculated as (Zhang et al. 2006):

$$D_{inner} = 0.363 \left( \frac{Q_{m}}{\rho} \right)^{0.45} \rho^{0.13} \mu^{0.025}$$  \hspace{1cm} (8)

where $D_{inner}$ is the inner diameter of the pipeline ($m$); $Q_{m}$ is the CO$_2$ mass flow rate in the pipeline ($kg/s$); $\rho$ is the CO$_2$ density at average temperature along the pipeline ($kg/m^3$); $\mu$ is the average CO$_2$ viscosity ($Pa\cdot s$).

For CO$_2$ pipeline, the average temperature, $T_{ave}$, is assumed to be the soil temperature, that is, $T_{ave} = T_{soil}$ (McCoy et al. 2008). Pressure and temperature affect the density and viscosity. The current research shows that the change of average temperature (soil temperature) is bounded along the buried pipeline (Zhang et al. 2012). This change is small, in this case, the density and viscosity are almost linear with pressure variation (NIST). The changes of density and viscosity caused by the variable temperature can be dealt with as the system uncertainties.

Based on the data from National Institute of Standards and Technology (NIST), (8) can be converted into:

$$D_{inner} = 0.363Q_{m}^{0.45} f_{\rho}(P_{ave}, T_{ave})^{0.32} f_{\mu}(P_{ave}, T_{ave})^{0.025}$$  \hspace{1cm} (9)

where $P_{ave}$ is the average pressure along the pipeline ($MPa$); $T_{ave}$ is the soil temperature around the pipeline ($^\circ C$). $f_{\rho}(P_{ave}, T_{ave})$ is the function of density that depends on $P_{ave}$ and $T_{ave}$ ($kg/m^3$); $f_{\mu}(P_{ave}, T_{ave})$ is the function of viscosity that depends on $P_{ave}$ and $T_{ave}$ ($Pa\cdot s$).

$P_{ave}$ can be calculated as (Mohitpour et al. 2003):

$$P_{ave} = \frac{2}{3} \left( P_{inlet} + P_{outlet} - \frac{P_{inlet} P_{outlet}}{P_{inlet} + P_{outlet}} \right)$$  \hspace{1cm} (10)

The density is given as a function of average pressure and temperature along the pipeline:
The viscosity is given as a function of average pressure and temperature along the pipeline:

$$f_\mu(P_{\text{ave}}, T_{\text{ave}}) = (BT)^T P$$  \hspace{1cm} (11)$$

$$f_\mu(P_{\text{ave}}, T_{\text{ave}}) = (DT)^T P$$  \hspace{1cm} (12)$$

where $B$ and $D$ are known constant matrices; $P$ is the matrix of $P_{\text{ave}}$; $T$ is the matrix of $T_{\text{ave}}$.

By using (9-12), (8) can be re-written as:

$$D_{\text{inner}} = 0.363 Q_{\text{ac}}^{0.45} \left( (BT)^T P \right)^{0.32} \left( (DT)^T P \right)^{0.025}$$  \hspace{1cm} (13)$$

The pipe wall thickness is given as (Chandel et al. 2010):

$$t = \frac{P_{\text{max}} D_{\text{inner}}}{2(S \cdot F \cdot E - P_{\text{max}})}$$  \hspace{1cm} (14)$$

where $t$ is the wall thickness (m); $P_{\text{max}}$ is the maximum allowable operation pressure ($MP_a$); $S$ is the specified minimum yield stress for the pipe material ($MP_a$); $E$ is the longitudinal joint factor; $F$ is the design factor.

Liquid pipeline transportation is researched in this study. Compared with supercritical fluid transportation, liquid transportation is better energy efficiency and lower transportation cost over long distance (Zhang et al. 2006; Zhang et al. 2012; Knoope et al. 2014). The pressure drop is calculated for all liquid cases as follows (Knoope et al. 2014):

$$\Delta P_{\text{ac}} = \frac{8 f Q_{\text{ac}}^2}{\rho_{\text{ac}} \pi^2 D_{\text{inner}}^2}$$  \hspace{1cm} (15)$$

where $f$ is the Darcy-Weisbach friction; $\rho_{\text{ac}}$ is the actual density along the pipeline ($kg/m^3$), it changes with the pressure drop and soil temperature, the change is dealt with as one of the multiple uncertainties.
In this paper, based on the levelized transport cost, the installation of boosting pump stations is an optimization design resulting from tradeoffs between increasing the inlet pressure, enlarging the pipeline diameter, or adding a boosting pump station. The number of boosting pump stations is calculated by (Knoope et al. 2014):

$$L_{pum} = \frac{P_{\text{inlet}} - P_{\text{out}}}{\Delta P_{\text{act}}}$$ (16)

$$N_{\text{pump}} = \left\lfloor \frac{L}{L_{pum}} \right\rfloor$$ (17)

where $L_{pum}$ is the maximum distance between the boosting pump stations; $N_{\text{pump}}$ is the number of boosting pump stations, $\left\lfloor \cdot \right\rfloor$ means the largest integer not greater than the enclosed ratio.

CO$_2$ velocity can be calculated as:

$$V = \frac{4Q_{\text{act}}}{\rho_{\text{act}} \times \pi \times D_{\text{inner}}^2}$$ (18)

where $V$ is the actual velocity ($m/s$), $\rho_{\text{act}}$ is the actual density along the pipeline.

3 Robust and stepwise optimization methods

In this Section, the robust optimization issue is solved by using LMI. Combined with robust and stepwise methods, the pipeline transportation optimization design algorithms are presented.

**Theorem 1:** Considering a robust optimization problem (7) with the uncertainties $u$, if there exist the decision variable $P_{\text{inlet}}$, auxiliary variables $\lambda$ and $\tau_j$ such that

$$\begin{bmatrix} \lambda + F(x) - \sum_{j=1}^{J} \tau_j e^2 & (N(x) + G)^T & E^T(x) \\ N(x) + G & \sum_{j=1}^{J} \tau_j Q_j & M^T(x) \\ E(x) & M(x) & I \end{bmatrix} \geq 0$$

$$P_{\text{outlet}} < P_{\text{inlet}}$$

$$P_{\text{inlet}} < P_{\text{max}}$$

$$V_{\text{min}} < V$$

$$V < V_{\text{max}}$$

$$P_{\text{out}} = P_{\text{inlet}} - \Delta P_{\text{act}} L / (N_{\text{pump}} + 1)$$

$$\tau_j \geq 0, \quad j = 1, \ldots, J$$

(19)
(7) can be transformed into an optimization problem that will be an objective function $\lambda$ with constraint (19). Some variables are defined as follows: $E(x) = A_x P_{\text{inlet}}$, $F(x) = -(2P_{\text{inlet}}^r B_0 + C_0)$, $M(x) = \begin{bmatrix} A_{P\text{inlet}} & A_{P\text{inlet}} & \cdots \end{bmatrix}$, $N(x) = \begin{bmatrix} P_{\text{inlet}}^r B_1^r \\ \vdots \\ P_{\text{inlet}}^r B_L^r \end{bmatrix}$, $G = -\frac{1}{2} \begin{bmatrix} C_1 \\ \vdots \\ C_L \end{bmatrix}$. According to (19), the pipeline robust optimization problem can be solved by using LMI. The readers can find the proof for Theorem 1 in Appendix A.

Combined with the proposed robust optimization approach, a stepwise optimization method is presented for designing the pipeline transportation, which can be divided into two steps: (1) The robust optimization of inner diameter and wall thickness; (2) The re-robust optimization of inlet pressure and number of boosting bump stations.

**Algorithm 1: The first step optimization**

**Step 1:** Selecting the minimum operational temperature and inlet pressure as the initial values of $T_{\text{ave}}$ and $P_{\text{inlet}}$, respectively.

**Step 2:** Substituting $T_{\text{ave}}$ and $P_{\text{inlet}}$ into (13) to compute $D_{\text{inner}}$; Substituting $D_{\text{inner}}$ into (14) to compute $t$.

**Step 3:** Substituting $D_{\text{inner}}$, $T_{\text{ave}}$ and $P_{\text{inlet}}$ into (18) to compute $V$. If $V_{\text{min}} < V < V_{\text{max}}$, then go to next step, else if, letting $P_{\text{inlet}} = P_{\text{inlet}} + \Delta P_{\text{inlet}}$ and go to Step 2.

**Step 4:** Substituting $D_{\text{inner}}$ into (17) to compute $N_{\text{pump}}$. If $N_{\text{pump}} \leq N_{\text{boost}}$, then go to next step, else if, letting $P_{\text{inlet}} = P_{\text{inlet}} + \Delta P_{\text{inlet}}$ and go to Step 2.

**Step 5:** Substituting all known parameters into (2) to get $LC$. If $P_{\text{inlet}} < P_{\text{max}}$, letting $P_{\text{inlet}} = P_{\text{inlet}} + \Delta P_{\text{inlet}}$, and go to Step 2, else if go to the next step.

**Step 6:** By using the enumeration method, comparing all the computed $LC$ and selecting the minimum one as $MLC$. Then computing $D_{\text{inner}}$, $t$ and $N_{\text{pump}}$ according to $MLC$. 
Step 7: If $T_{av} < T_{maxop}$, then letting $T_{av} = T_{av} + \Delta T_{av}$, $P_{inlet} = P_{min}$ and go to Step 2, else if go to next step.

Step 8: Dividing the optimal $MLC$ function with different ‘pieces’ and representing each ‘piece’ with linear function; Obtaining $A_0$, $\delta_a$, $B_0$, $\delta_b$, $C_0$, $\delta_c$, establishing the robust optimization model (7).

Step 9: Substituting the parameters of each ‘piece’ into (19), obtaining the robust optimization results by using LMI toolbox, selecting the smallest one.

Step 10: Calculating the wall thickness based on (14); Obtaining inner diameter of the NPS ($ID_{NPS}$) and wall thickness of the NPS ($t_{NPS}$) by selecting from the NPS.

End the program.

For the second step optimization, Algorithm 2 calculates the final $P_{inlet}$ and $N_{pump}$.

**Algorithm 2: The second step optimization**

Step 1: Selecting the minimum operational temperature and inlet pressure as the initial values of $T_{av}$ and $P_{inlet}$, respectively.

Step 2: Substituting $ID_{NPS}$, $t_{NPS}$, $T_{av}$ and $P_{inlet}$ into (18) to compute $V$. If $V_{min} < V < V_{max}$, then go to next step, else if, letting $P_{inlet} = P_{inlet} + \Delta P_{inlet}$ and go to Step 2.

Step 3: Substituting $ID_{NPS}$, $t_{NPS}$ into (17) to calculate $N_{pump}$. If $N_{pump} \leq N_{Boost}$, then go to next step, else if, letting $P_{inlet} = P_{inlet} + \Delta P_{inlet}$ and go to Step 2.

Step 4: Substituting all known parameters into (2) to get $LC$. If $P_{inlet} < P_{max}$, letting $P_{inlet} = P_{inlet} + \Delta P_{inlet}$, and go to Step 2, else if go to the next step.

Step 5: By using the enumeration method, comparing all the computed $LC$ and selecting the minimum one as $MLC$.

Step 6: If $T_{av} < T_{maxop}$, then letting $T_{av} = T_{av} + \Delta T_{av}$, and go to Step 2, else if go to next step.

Step 7: Dividing the optimal $MLC$ function with different ‘pieces’ and representing each ‘piece’ with linear
function; Obtaining $A_0$, $\delta_{A_0}$, $B_0$, $\delta_{B_0}$, $C_0$, $\delta_{C_0}$, establishing the robust optimization model (7).

Step 8: Substituting the parameters of each ‘piece’ into (19), obtaining the robust optimization results by using LMI toolbox, getting the optimal $LC$ and related $P_{\text{inlet}}$, $N_{\text{pump}}$.

End the program.

After the executing the above algorithms, the optimal $ID_{\text{XPS}}$ and $t_{\text{XPS}}$ can be obtained from the Step 10 of Algorithm 1, the optimal $P_{\text{inlet}}$, $N_{\text{pump}}$ can be obtained from the Step 8 of Algorithm 2.

Remark 2: Algorithm 1 and 2 are used to deal with the variable temperature, if considering the other multiple uncertainties, these uncertainties can be considered as perturbations of the nominal parameters.

4 Computation results and analysis

The basic parameters of the transportation are given in Table 2. The other detailed parameters are given in Table 3-5.

Table 2. Basic parameters of the transportation (Chandel et al. 2010; Gao et al. 2011; Zhang et al. 2012)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soil temperature ($^\circ\text{C}$)</td>
<td>$T_{\text{soil}}$</td>
<td>2~17</td>
</tr>
<tr>
<td>CO$_2$ inlet pressure (MPa)</td>
<td>$P_{\text{inlet}}$</td>
<td>8.6~15.3</td>
</tr>
<tr>
<td>Pipeline length (km)</td>
<td>$L$</td>
<td>500</td>
</tr>
<tr>
<td>Injection pressure (MPa)</td>
<td>$P_{\text{inject}}$</td>
<td>10</td>
</tr>
<tr>
<td>Operation time (hour)</td>
<td>$H_{\text{ope}}$</td>
<td>8760</td>
</tr>
</tbody>
</table>

Table 3. Detailed parameter values of pipeline (McCoy et al. 2008; Vandeginste et al. 2008)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specified minimum yield stress for X70 steel (MPa)</td>
<td>$S$</td>
<td>483</td>
</tr>
<tr>
<td>Longitudinal joint factor</td>
<td>$E$</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Table 4. Detailed parameter values of compressor and boosting pump stations (Zhang et al. 2006; Kuramochi et al. 2012; Knoope et al. 2014)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Universal gas constant (J/mol K)</td>
<td>$R$</td>
<td>8.3145</td>
</tr>
<tr>
<td>Suction temperature (K)</td>
<td>$T_1$</td>
<td>313.15</td>
</tr>
<tr>
<td>Specific heat ratio ($c_p/c_v$)</td>
<td>$\gamma$</td>
<td>1.294</td>
</tr>
<tr>
<td>Molar mass (g/mol)</td>
<td>$M$</td>
<td>44.01</td>
</tr>
<tr>
<td>Total number of compression stages</td>
<td>$N$</td>
<td>4</td>
</tr>
<tr>
<td>Isentropic efficiency</td>
<td>$\eta_{iso}$</td>
<td>80%</td>
</tr>
<tr>
<td>Mechanical efficiency</td>
<td>$\eta_{mech}$</td>
<td>99%</td>
</tr>
<tr>
<td>Suction pressure (MPa)</td>
<td>$P_{cap}$</td>
<td>0.101</td>
</tr>
<tr>
<td>Discharge pressure (MPa)</td>
<td>$P_{MOP}$</td>
<td>8.6</td>
</tr>
<tr>
<td>Base costs for calculating the compressor capital cost (Me€)</td>
<td>$I_0$</td>
<td>21.9</td>
</tr>
<tr>
<td>Base scale of the compressor (MW e)</td>
<td>$W_{comp,e}$</td>
<td>13</td>
</tr>
<tr>
<td>Scaling factor</td>
<td>$\gamma$</td>
<td>0.67</td>
</tr>
<tr>
<td>Multiplication exponent</td>
<td>$n$</td>
<td>0.9</td>
</tr>
<tr>
<td>Percentage of the capital cost of booster pumps</td>
<td>$f_{BOP,M}$</td>
<td>0.04</td>
</tr>
<tr>
<td>Booster pump efficiency</td>
<td>$\eta_{booster}$</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Operation time of compressor (hour) \( T_C \) 8760

Operation time of booster pump (hour) \( T_B \) 8760

Price of electricity (€/per kilowatt hour) \( C_{kw} \) 0.0437

Number of boosting pump stations \( N_{pump} \) \( \leq 5 \)

Actual velocity (m/s) \( V \) \( 0.5 < V < 6 \)

---

**Table 5. Parameter values of the levelized cost model (Knoope et al. 2013; Knoope et al. 2014)**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount rate (%)</td>
<td>( r )</td>
<td>15</td>
</tr>
<tr>
<td>Design lifetime of the pipeline (years)</td>
<td>( z_1 )</td>
<td>50</td>
</tr>
<tr>
<td>Design lifetime of compressors (years)</td>
<td>( z_2 )</td>
<td>25</td>
</tr>
<tr>
<td>Design lifetime of the booster pumps (years)</td>
<td>( z_3 )</td>
<td>25</td>
</tr>
</tbody>
</table>

---

**Table 6 Robust optimization results for different design mass flow rates**

<table>
<thead>
<tr>
<th>Mass flow rate ( Q_m ) (kg/s)</th>
<th>120</th>
<th>150</th>
<th>200</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_{inlet} ) (MPa)</td>
<td>13.7717</td>
<td>13.6839</td>
<td>13.2998</td>
<td>13.0346</td>
</tr>
<tr>
<td>( ID_{NPS} ) (m)</td>
<td>0.3115</td>
<td>0.33975</td>
<td>0.39055</td>
<td>0.44135</td>
</tr>
<tr>
<td>( t_{NPS} ) (m)</td>
<td>0.00635</td>
<td>0.007925</td>
<td>0.007925</td>
<td>0.007925</td>
</tr>
<tr>
<td>Total cost ( (\text{e}) )</td>
<td>1,290,803,170~</td>
<td>1,659,755,448~</td>
<td>2,024,327,376~</td>
<td>2,383,628,850~</td>
</tr>
<tr>
<td>The second step</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_{inlet} ) (MPa)</td>
<td>12.3079</td>
<td>12.2640</td>
<td>11.9307</td>
<td>11.5827</td>
</tr>
<tr>
<td>Total cost ( (\text{e}) )</td>
<td>1,282,572,274~</td>
<td>1,649,762,478~</td>
<td>2,011,444,920~</td>
<td>2,366,520,570~</td>
</tr>
</tbody>
</table>
Table 6 shows the robust optimization results for different design mass flow rates with variable temperature, compared with the first step optimization results, it can be seen that the second step saves the cost. To further illustrate this advantage, the full saving costs are given in Figure 1 for lifetime 25 years, note that full saving cost is the first step total cost minus the second step total cost. For instance, the design mass flow rate is 120 kg/s, the full saving costs are 8,230,896~8,902,613 € for 25 years. The reasons are given as: In the first step, the optimal ideal inner diameter (ID) and wall thickness (t) are computed by using the given design conditions. However, the final selected inner diameter and wall thickness are the NPS in the engineering practice which are larger than the ideal ones. Based on the first step optimization, the second stepwise can re-optimize the inlet pressure and the number of boosting pump stations, which can improve the optimal performance.

Table 7 shows the robust optimization results with multiple uncertainties. The flow rate is assumed to be 130 kg/s.
The CO₂ temperature is variable with the seasons. The other multiple uncertainties are considered as perturbations, which are denoted as the random percentages ($\delta_p$) of the nominal parameters. It can be seen that the levelized cost increases with the uncertainty increases.

Table 7 Robust optimization results with multiple uncertainties

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\delta_p \in$ [-0.25%,0.25%]</th>
<th>[-0.5%,0.5%]</th>
<th>[-0.75%,0.75%]</th>
<th>[-1%,1%]</th>
<th>[-1.5%,1.5%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ID_{NPS}$ (m)</td>
<td>0.31115</td>
<td>0.31115</td>
<td>0.31115</td>
<td>0.31115</td>
<td>0.31115</td>
</tr>
<tr>
<td>$t_{NPS}$ (m)</td>
<td>0.00635</td>
<td>0.00635</td>
<td>0.00635</td>
<td>0.00635</td>
<td>0.00635</td>
</tr>
<tr>
<td>$P_{inlet}$ (MPa)</td>
<td>12.6971</td>
<td>12.6969</td>
<td>12.6966</td>
<td>12.6951</td>
<td>12.6890</td>
</tr>
</tbody>
</table>

To further illustrate the proposed approach, it will be compared with the existing methods. Two situations are presented as follows: (1) Considering the temperature uncertainty only (2) Not only considering the variable temperature but also the other multiple uncertainties.

Table 8 Comparison results of the existing and proposed methods with variable temperature

<table>
<thead>
<tr>
<th>Method</th>
<th>Parameter</th>
<th>$Q_w = 123$ kg/s</th>
<th>$Q_w = 170$ kg/s</th>
<th>$Q_w = 200$ kg/s</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P_{inlet}$ (MPa)</td>
<td>$L = 500$ km</td>
<td>$L = 380$ km</td>
<td>$L = 470$ km</td>
</tr>
<tr>
<td>The existing</td>
<td>$ID_{NPS}$ (m)</td>
<td>0.31115</td>
<td>0.33975</td>
<td>0.39055</td>
</tr>
<tr>
<td>method (Chandel et al. 2010)</td>
<td>$t_{NPS}$ (m)</td>
<td>0.00635</td>
<td>0.007925</td>
<td>0.007925</td>
</tr>
<tr>
<td></td>
<td>Total cost (€)</td>
<td>1,301,231,319~</td>
<td>1,561,273,948~</td>
<td>1,964,157,458~</td>
</tr>
</tbody>
</table>
Assuming the temperature is variable with seasons. The design should satisfy the following constraint:

\[ P_{\text{out}} = P_{\text{inlet}} - \Delta P_{\text{act}} L_f / (N_{\text{pump}} + 1) \]

It is important to note that \( P_{\text{out}} = 10 \) MPa is the minimum injection pressure (Zhang et al. 2012). The CO₂ temperature is assumed to be 12 °C by using the existing method (Chandel et al. 2010). Table 8 shows the comparison results of the existing and proposed methods with variable temperature. It can be seen that \( P_{\text{out}} \) may not satisfy the constraint based on the existing method, compared with the existing method, the proposed method saves the total cost. For example, assuming \( Q_m = 123 \text{ kg/s} \) and \( L = 500 \text{ m} \), based on the existing method, the inlet pressure is 13 MPa, the optimized nominal inner diameter and wall thickness are 0.31115 m and 0.00635 m respectively, \( P_{\text{out}} \) decreases from 10.15900 to 9.90296 MPa as the temperature increases from 2~17 °C. Therefore, if the optimization design is applied based on the existing method, \( P_{\text{out}} \) is smaller than 10 MPa at higher temperatures, this lead to an infeasible design. Based on the proposed approach, \( P_{\text{out}} \) decreases from 10.20830 to 10 MPa as the temperature increases. The proposed method satisfy the constraint. That’s because the existing optimization design based on a constant temperature between the variable soil temperature, which ignore the effects of variable temperature. Over the life time of 25

<table>
<thead>
<tr>
<th>Method</th>
<th>( P_{\text{inlet}} ) (MPa)</th>
<th>( \Delta P ) (MPa)</th>
<th>( N_{\text{pump}} )</th>
<th>( L_f ) (m)</th>
<th>( Q_m ) (kg/s)</th>
<th>( T_f ) (°C)</th>
<th>( T_r ) (°C)</th>
<th>( P_{\text{out}} ) (MPa)</th>
<th>( \Delta T ) (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Existing</td>
<td>10.20830~10</td>
<td>10.18282~10</td>
<td>10.15295~10</td>
<td>10.20830~10</td>
<td>10.18282~10</td>
<td>10.15295~10</td>
<td>10.20830~10</td>
<td>10.18282~10</td>
<td>10.15295~10</td>
</tr>
</tbody>
</table>

Table 8 shows the comparison results of the existing and proposed methods with variable temperature. It can be seen that \( P_{\text{out}} \) may not satisfy the constraint based on the existing method, compared with the existing method, the proposed method saves the total cost. For example, assuming \( Q_m = 123 \text{ kg/s} \) and \( L = 500 \text{ m} \), based on the existing method, the inlet pressure is 13 MPa, the optimized nominal inner diameter and wall thickness are 0.31115 m and 0.00635 m respectively, \( P_{\text{out}} \) decreases from 10.15900 to 9.90296 MPa as the temperature increases from 2~17 °C. Therefore, if the optimization design is applied based on the existing method, \( P_{\text{out}} \) is smaller than 10 MPa at higher temperatures, this lead to an infeasible design. Based on the proposed approach, \( P_{\text{out}} \) decreases from 10.20830 to 10 MPa as the temperature increases. The proposed method satisfy the constraint. That’s because the existing optimization design based on a constant temperature between the variable soil temperature, which ignore the effects of variable temperature. Over the life time of 25
years, the optimal total costs are 1,301,231,319–1,303,784,612 and 1,298,899,850–1,300,568,495 € by using the existing and proposed methods, respectively. The proposed method saves 2,331,469–3,216,117 €. The proposed method not only satisfies the constraint but also saves total cost. Therefore, the optimal results are more reasonable by using the proposed approach.

Table 9 (a) Comparison results of the existing and proposed methods with multiple uncertainties

<table>
<thead>
<tr>
<th>Method</th>
<th>Parameter</th>
<th>$Q_n$ (kg/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>The existing research</td>
<td>$ID_{NGS}$ (m)</td>
<td>0.26035</td>
</tr>
<tr>
<td>(Knoope et al. 2014)</td>
<td>$t_{NGS}$ (m)</td>
<td>0.00635</td>
</tr>
<tr>
<td></td>
<td>$P_{sia}$ (MPa)</td>
<td>14.1287</td>
</tr>
<tr>
<td>The proposed method</td>
<td>$ID_{NGS}$ (m)</td>
<td>0.31115</td>
</tr>
<tr>
<td></td>
<td>$t_{NGS}$ (m)</td>
<td>0.00635</td>
</tr>
<tr>
<td></td>
<td>$P_{sia}$ (MPa)</td>
<td>11.6051</td>
</tr>
<tr>
<td>Base optimal results with</td>
<td>$ID_d$ (m)</td>
<td>0.26306</td>
</tr>
<tr>
<td>known uncertainties</td>
<td>$t_d$ (m)</td>
<td>0.00563</td>
</tr>
<tr>
<td></td>
<td>$P_d$ (MPa)</td>
<td>14.2718</td>
</tr>
</tbody>
</table>

Assuming the temperature is variable with seasons, other multiple uncertainties are bounded ($\delta_p \in [-3\%, 3\%]$).

In order to further illustrate the effects for dealing with multiple uncertainties by using the proposed method, known perturbations are given as basic reference: the variable temperature is $17\ ^\circ\text{C}$, other multiple uncertainties are 2% of nominal parameters, $ID_d$ and $t_d$ can be obtained. Table 9 (a) shows the optimization results of the existing and proposed methods with multiple uncertainties. Compared with the results from the basic reference, diameter and wall thickness not satisfy the design by using the existing method (Knoope et al. 2014). For example,
assuming \( Q_m = 100 \text{ kg/s}, \) \( ID_{NPS} = 0.26035 \text{ m} \) is obtained by using the existing method, compared with \( ID_d = 0.26306 \text{ m}, \) \( ID_{NPS} \) cannot satisfy the diameter design requirement. Assuming \( Q_m = 145 \text{ kg/s}, \) \( t_{NPS} = 0.00635 \text{ m} \) is obtained by using the existing method, compared with \( t_d = 0.00647 \text{ m}, \) \( t_{NPS} \) cannot satisfy the wall thickness design requirement, there is no safety guarantee for the pipeline transportation.

The existing method may make an infeasible design, because it cannot deal with the effects caused by the variable temperature and other multiple uncertainties effectively. Note that the determination of diameter and wall thickness depends on temperature indeed (as shown in (13) and (14)). Compared with the results from the basic reference, the proposed method can deal with the multiple uncertainties well and get feasible results.

Table 9 (b) Comparison results of the existing and proposed methods with multiple uncertainties

<table>
<thead>
<tr>
<th>Method</th>
<th>Parameter</th>
<th>( Q_m ) (kg/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>120</td>
</tr>
<tr>
<td>The existing research</td>
<td>( ID_{NPS} ) (m)</td>
<td>0.31115</td>
</tr>
<tr>
<td></td>
<td>( t_{NPS} ) (m)</td>
<td>0.00635</td>
</tr>
<tr>
<td></td>
<td>( P_{NPS} ) (MPa)</td>
<td>13.9803</td>
</tr>
<tr>
<td></td>
<td>Total cost (€)</td>
<td>1,335,281,502</td>
</tr>
<tr>
<td>The proposed method</td>
<td>( ID_{NPS} ) (m)</td>
<td>0.31115</td>
</tr>
<tr>
<td></td>
<td>( t_{NPS} ) (m)</td>
<td>0.00635</td>
</tr>
<tr>
<td></td>
<td>( P_{NPS} ) (MPa)</td>
<td>12.3098</td>
</tr>
<tr>
<td></td>
<td>Total cost (€)</td>
<td>1,325,140,505</td>
</tr>
<tr>
<td></td>
<td>Total saving (€)</td>
<td>10,140,997</td>
</tr>
</tbody>
</table>

Table 9 (b) also shows the comparison results of the existing and proposed methods with multiple uncertainties. Compared with the existing method, the proposed method saves the total cost. For example, assuming \( Q_m = 120 \)
kg/s, the optimal $\bar{ID}_{NPS} = 0.31115$ and $t_{NPS} = 0.00635$ m are obtained by using the two methods. The optimal total costs are 1,335,281,502 and 1,325,140,505 € over the lifetime of 25 years for the existing and proposed methods, respectively. The proposed method saves 10,140,997 €, which improves the optimization performance.

5 Conclusion

In order to minimize LC for pipeline design, a novel robust optimization model is developed by considering multiple uncertainties. The solution for robust optimization problem is obtained by LMI. A stepwise optimization is given to improve the optimization performance. In the numerical studies, comparing with the existing optimization methods, it is verified that the proposed approach can improve the design performance and provides more securities for the pipeline transportation. In the future, the authors will focus on the applications of the proposed approach in the CO$_2$ pipeline design.

Acknowledgments

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Appendix A: Proof for Theorem 1

**Lemma 1** (El Ghaoui 1997): (S-procedure). Letting $F_0, \ldots$ be quadratic functions of the variable $\zeta \in \mathbb{R}^n$:

$$F_i(\zeta) \geq 0 \quad \text{for all } \zeta \quad \text{such that} \quad F_i(\zeta) \geq 0, \quad i = 1, \ldots$$

where $T_i = T_i^T$. If $F_0, \ldots$ satisfies the following condition:
there are $\tau_i \geq 0, \ldots$ such that

$$\begin{bmatrix} T_0 & e_0^T \\ e_0 & v_0 \end{bmatrix} - \sum_{i=1}^p \tau_i \begin{bmatrix} T_i & e_i^T \\ e_i & v_i \end{bmatrix} \geq 0.$$  

**Lemma 2** (Lin C 2007) (Schur complement) Letting $S_1$, $S_2$, and $S_3$ be appropriately dimensional matrices with $S_1$ and $S_3$ symmetric. Then,

$$\begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} > 0$$

if and only if any of the following conditions holds:

(i) $S_1 > 0$ and $S_3 - S_2^T S_1^{-1} S_2 > 0$;

(ii) $S_3 > 0$ and $S_1 - S_2 S_3^{-1} S_2^T > 0$.

**Proof:** Introducing an auxiliary variable $\lambda$, (7) can be reformed as:

$$\min_{\mathbf{P}_{\text{act}}} \max\left\{ \lambda \mathbf{H} \right\} \text{ s.t. } \begin{align*}
\mathbf{P}_{\text{inlet}} & \sim \mathbf{P}_{\text{max}} \sim \mathbf{P}_{\text{max}} \\
\mathbf{P}_{\text{outlet}} & < \mathbf{P}_{\text{inlet}} \\
\mathbf{P}_{\text{inlet}} & < \mathbf{P}_{\text{max}} \\
V_{\text{min}} & < V \\
V & < V_{\text{max}} \\
\mathbf{P}_{\text{out}} & = \mathbf{P}_{\text{inlet}} - \Delta P_{\text{act}} L / (N_{\text{pump}} + 1)
\end{align*}$$  

(20)

Defining the following norm:

$$\|u\|_{Q_j} = \sqrt{u^T Q_j u} \quad j = 1, 2, \ldots$$  

(21)

where $Q_j \geq 0$, $\sum_{j=1}^J Q_j > 0$. Setting $Q_j$ be an appropriately dimensional matrix, the element of the $j$th row and $j$th column of $Q_j$ is 1, while the rest elements of $Q_j$ are 0. Then $\|u\|_{Q_j} = \|u_j\|_{Q_j}$, $j = 1, 2, \ldots$. $\|u\|_{\mathcal{H}} \leq \varepsilon$ can be denoted as $|u_j| \leq \varepsilon$, $j = 1, 2, \ldots$. Therefore, $\|u\|_{\mathcal{H}} \leq \varepsilon$ can be written as $\|u\|_{Q_j} \leq \varepsilon$, $j = 1, 2, \ldots$. (20) can be transformed into:
min \( \lambda \)

\[ \begin{aligned}
\text{s.t.} & \quad P_{\text{inlet}} \leq P_{\text{max}} \\
& \quad V_{\text{min}} < V \\
& \quad V < V_{\text{max}} \\
& \quad P_{\text{out}} = P_{\text{inlet}} - \Delta P_{\text{act}} L / (N_{\text{pump}} + 1)
\end{aligned} \] (22)

\( u^T Q u \leq \varepsilon^2 \) is equivalent to:

\[
\begin{bmatrix}
1^T \\
u
\end{bmatrix}
\begin{bmatrix}
\varepsilon^2 & 0 \\
0 & -Q_j
\end{bmatrix}
\begin{bmatrix}
1 \\
u
\end{bmatrix} \geq 0
\] (23)

\( P_{\text{inlet}} \) can be written as:

\[
P_{\text{inlet}}^T \left( A_0 + \sum_{i=1}^L u_i A_i \right) P_{\text{inlet}} + \left( B_0 + \sum_{i=1}^L u_i B_i \right) P_{\text{inlet}} + \left( C_0 + \sum_{i=1}^L u_i C_i \right) \leq \lambda (24)
\]

Defining variables transformation, \( E(x) = A_0 P_{\text{inlet}} \), \( F(x) = -\left( 2P_{\text{inlet}}^T B_0 + C_0 \right) \),

\[
\begin{bmatrix}
E(x) + M(x)u \\
J
\end{bmatrix}
\begin{bmatrix}
E(x) + M(x)u \\
N(x) + G - \varepsilon^2
\end{bmatrix} \leq 2\left( N(x) + G \right)^T u + F(x) + \lambda
\] (25)

After straight forward manipulations, (25) becomes:

\[
\begin{bmatrix}
1^T \\
u
\end{bmatrix}
\begin{bmatrix}
\lambda + F(x) - E(x)^T E(x) \\
N(x) + G - \varepsilon^2
\end{bmatrix} \begin{bmatrix}
E(x) + M(x) \\
M(x)
\end{bmatrix} \geq 0
\] (26)

Using the S-procedure (Lemma 1), for all \( u \), (26) holds if there exist a scalar \( \tau_j \geq 0 \) such that

\[
\begin{bmatrix}
\lambda + F(x) - E(x)^T E(x) \\
N(x) + G - \varepsilon^2
\end{bmatrix} \begin{bmatrix}
E(x) + M(x) \\
M(x)
\end{bmatrix} \geq 0
\] (27)

After straight forward manipulations, (27) is equivalent to:

\[
\begin{bmatrix}
\lambda + F(x) - \sum_{j=1}^L \tau_j \varepsilon^2 \\
N(x) + G
\end{bmatrix} \begin{bmatrix}
E(x) + M(x) \\
E(x) + M(x)
\end{bmatrix} \geq 0
\] (28)
Using Schur complement (Lemma 2), (28) is transformed into:

\[
\begin{bmatrix}
\lambda + F(x) & -\sum_{j=1}^{J} \tau_j c \cdot x

(N(x) + G_j)^T E(x)

N(x) + G_i & \sum_{j=1}^{J} \tau_j Q_j M^T(x)

E(x) & M(x) I
\end{bmatrix} \geq 0
\]

(29)

According to (19), the pipeline robust optimization problem can be solved by using LMI.

Reference


